

Arboricity and spanning-tree packing of random graphs

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Abstract

We study the arboricity A and the maximum number T of edge-disjoint spanning trees of the Erdős-Rényi random graph $\mathcal{G}(n, p)$. For all $p \in [0, 1]$, we show that, with high probability, T is precisely the minimum between δ and $\lfloor m/(n-1) \rfloor$, where δ is the smallest degree of the graph and m denotes the number of edges. Moreover, we explicitly determine a threshold value for p such that: above this threshold, T equals $\lfloor m/(n-1) \rfloor$ and A equals $\lceil m/(n-1) \rceil$; and below this threshold, T equals δ , and we give a two-value concentration result for the arboricity A in that range. Finally, we include a stronger version of these results in the context of the random graph process where the edges are sequentially added one by one.

1 Introduction

The spanning-tree packing (STP) number of a connected graph is the maximum number of edge-disjoint spanning trees it contains. Computing this parameter is a very classical problem in combinatorial optimization. One of the earliest results on the STP number is a min-max relation proved by Tutte [26] and Nash-Williams [20]: the STP number of a graph is the minimum value, ranging over all partitions \mathcal{P} of the vertex set, of the ratio (rounded down) between the number of edges across \mathcal{P} (i.e. edges with ends lying in different classes of \mathcal{P}) and $|\mathcal{P}| - 1$. This characterisation has important consequences in computer science, where the STP number has been used as a measure of network vulnerability in case of attack or edge failure (see [12, 8]). Intuitively speaking, it provides information about the number of edges that must be destroyed in a connected network in order to create a given number of new components. In addition, finding edge-disjoint spanning trees in a graph is relevant to the design of efficient and robust communication protocols (see e.g. a seminal article by Itai and Rodeh [14]). There are two obvious upper bounds on the STP number of a graph with n vertices: the minimum degree, since each spanning tree would need to use at least one edge incident to each vertex; and the number of edges divided by $n - 1$, since each spanning tree has exactly $n - 1$ edges. For further information on the STP number, we refer the reader to a survey by Palmer [22] on this topic.

Another closely related graph parameter that has been widely studied is the arboricity of a graph: the minimum number of subforests needed to cover all of its edges. A trivial lower

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bound on the arboricity of a graph with n vertices is the number of edges divided by $n - 1$, since we cannot do better than covering all the edges with a set of edge-disjoint spanning trees. Nash-Williams [21] also provided a min-max relation for the arboricity of a graph, which yields a natural interpretation of arboricity as a measure of density of the subgraphs of a graph. This makes arboricity a useful notion in computer science, since the problem of determining the existence of dense subgraphs in large graphs is relevant to many applications in real world domains like social networking or internet computing. In fact, finding such dense subgraphs and other related problems can often be efficiently solved in linear time for any class of graphs with bounded arboricity (see [7, 11]). This includes important families of graphs such as all minor-closed classes (e.g. planar graphs and graphs with bounded treewidth) and random graphs generated by the preferential attachment model. Finding the STP number and the arboricity of a given graph are among the most successful applications of matroids in combinatorial optimization. Both problems can be formulated as matroid union problems and thus can be solved in polynomial time. For more details, see [25, Chapter 51].

It is then natural to study the behaviour of the STP number and the arboricity for the Erdős-Rényi random graph $\mathcal{G}(n, p)$, in which the vertex set is $[n]$ and each of the $\binom{n}{2}$ possible edges is included independently with probability p (where $p = p(n)$ is a function of n). It is a well-known fact that, for $p = (\log n - \omega(1))/n$, the random graph $\mathcal{G}(n, p)$ is a.a.s.¹ disconnected (see e.g. Theorem 7.3 in [2]), and hence the STP number is zero. Palmer and Spencer [23] showed that a.a.s. the STP number of $\mathcal{G}(n, p)$ equals the minimum degree whenever this has constant value k , which happens when p is around $(\log n + (k - 1) \log \log n + O(1))/n$. In fact, they proved a stronger hitting-time result in the context of the evolution of $\mathcal{G}(n, p)$ when p grows gradually from 0 to 1, and showed that a.a.s. the precise time when the minimum degree first becomes k coincides with the time when k edge-disjoint spanning trees first appear. Moreover, Catlin, Chen and Palmer [5] studied the denser case of $p = C(\log n/n)^{1/3}$, where $C > 0$ is a sufficiently large constant, and determined the STP number and the arboricity of $\mathcal{G}(n, p)$ to be a.a.s. equal to $\lfloor m/(n - 1) \rfloor$ and $\lceil m/(n - 1) \rceil$, respectively, where m denotes the number of edges. In a recent unpublished manuscript, Chen, Li and Lian [6] proved that, for any $(\log n + \omega(1))/n \leq p \leq 1.1 \log n/n$, a.a.s. the STP number of $\mathcal{G}(n, p)$ equals the minimum degree. They also observed that this property a.a.s. does not hold for $p \geq 51 \log n/n$, and posed the question of what is the smallest value of p such that the STP number of $\mathcal{G}(n, p)$ differs from the minimum degree.

In this paper we strengthen the previous results, and characterise the STP number and the arboricity of $\mathcal{G}(n, p)$ as follows. We first prove that for all $p \in [0, 1]$, the STP number is a.a.s. the minimum between δ and $\lfloor m/(n - 1) \rfloor$, where δ and m respectively denote the minimum degree and the number of edges of $\mathcal{G}(n, p)$ (see Theorem 1). Note that the quantities δ and $\lfloor m/(n - 1) \rfloor$ above correspond to the two trivial upper bounds observed earlier for arbitrary graphs, so this implies that we can a.a.s. find a best-possible number of edge-disjoint spanning trees in $\mathcal{G}(n, p)$. Our argument uses several properties of $\mathcal{G}(n, p)$ in order to bound the number of crossing edges between subsets of vertices with certain restrictions, and then applies the characterisation of the STP number by Tutte and Nash-Williams stated in Theorem 7. Moreover, we determine the ranges of p for which the STP number takes each of these two values: δ and $\lfloor m/(n - 1) \rfloor$. In spite of the fact that the property $\{\delta \leq \lfloor m/(n - 1) \rfloor\}$ is not necessarily monotonic with respect to p , we show that it has a sharp threshold at $p \sim \beta \log n/n$,

¹We say that a sequence of events H_n holds *asymptotically almost surely* (a.a.s.) if $\lim_{n \rightarrow \infty} \Pr(H_n) = 1$.

where $\beta \approx 6.51778$ is a constant defined in Theorem 2. Below this threshold, the STP number of $\mathcal{G}(n, p)$ is a.a.s. equal to δ ; and above the threshold it is a.a.s. $\lfloor m/(n-1) \rfloor$. In particular, this settles the question raised by Chen, Li and Lian [6]. We also include a stronger version of these results in the context of the random graph process in which p gradually grows from 0 to 1 (or, similarly, the edges are added one by one). This provides a full characterisation of the STP number that holds a.a.s. simultaneously during the whole random graph process (see Theorem 3). The argument combines a more accurate version of the same ideas used in the analysis of the STP number of $\mathcal{G}(n, p)$ together with multiple couplings of $\mathcal{G}(n, p)$ at different values of p . In addition, the article contains several results about the arboricity of $\mathcal{G}(n, p)$. As an almost direct application of our result on the STP number, for p above the threshold $\beta \log n/n$, we determine the arboricity of $\mathcal{G}(n, p)$ to be a.a.s. equal to $\lceil m/(n-1) \rceil$. This significantly extends the range of p in the result by Catlin, Chen and Palmer [5]. We further prove that for all other values of p , the arboricity of $\mathcal{G}(n, p)$ is concentrated on at most two values (see Theorem 4). In order to prove this for the case $pn \rightarrow \infty$, we add $o(n)$ edges to $\mathcal{G}(n, p)$ in a convenient way that guarantees a full decomposition of the resulting graph into edge-disjoint spanning trees. This construction builds upon some of the ideas previously used to study the STP number. The case $pn = O(1)$ uses different proof techniques which rely on the structure of the k -core of $\mathcal{G}(n, p)$ together with the Nash-Williams characterisation of arboricity stated in Theorem 8. Finally, some of the aforementioned results on the arboricity are also given in the more precise context of the random graph process (see Theorem 5 and Corollary 6), similarly as we did for the STP number.

The behaviour of the STP number and the arboricity has been also studied in other models of random graphs. Frieze and Luczak [10] considered the random directed graph in which each vertex chooses k out-neighbours uniformly at random, with fixed k . This graph has k disjoint spanning trees with probability going to 1 (where the orientation of the arcs is ignored). Some variants of arboricity have also been studied. The linear arboricity of a graph is the minimum number of forests consisting only of paths needed to cover all edges of the graph. This parameter was studied by McDiarmid and Reed [18] for regular graphs. Similarly, the star arboricity of a graph requires each forest to consist only of stars and was studied by Alon, McDiarmid and Reed [1].

2 Main results

Given a graph G , let $V(G)$ denote the vertex set of G and let $E(G)$ denote the edge set of G . Let $m(G)$ be the number of edges of G and let $\delta(G)$ denote the minimum degree of G and define $\bar{d}(G) := 2m(G)/(|V(G)| - 1)$. Note that $\bar{d}(G)$ differs from the average degree of G by a small factor of $|V(G)|/(|V(G)| - 1)$.

Let $\mathcal{G}(n, p)$ denote the random graph with vertex set $[n]$ such that each possible edge in $\{\{u, v\} : u, v \in [n], u \neq v\}$ is included independently with probability p . Given a sequence of events $(E_n)_{n \in \mathbb{N}}$, we say that E_n happens *asymptotically almost surely* (a.a.s.) if $\Pr(E_n) \rightarrow 1$ as $n \rightarrow \infty$.

Given real sequences a_n and b_n (possibly taking negative values), we write $a_n = O(b_n)$ if there is a constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all n ; we write $a_n = o(b_n)$ if $\lim_{n \rightarrow \infty} a_n/b_n = 0$. We write $a_n = \Omega(b_n)$ if $a_n \geq 0$ and $b_n = O(a_n)$; $a_n = \omega(b_n)$ if $a_n \geq 0$ and $b_n = o(a_n)$; $a_n = \Theta(b_n)$ if $a_n \geq 0$, $a_n = O(b_n)$ and $a_n = \Omega(b_n)$. In particular, in this paper,

all constants involved in these notations do not depend on p under discussion. For instance, if we have $a_n = \Omega(b_n)$, where b_n may be an expression involving $p = p(n)$, then it means that there are constants $C > 0$ and n_0 (both independent with p), such that $a_n \geq C|b_n|$ uniformly for all $n \geq n_0$ and for all p in the range under discussion.

For any graph G , let $T(G)$ denote the maximum number of edge-disjoint spanning trees in G (possibly 0 if G is disconnected).

Theorem 1. *For every $p = p(n) \in [0, 1]$, we have that a.a.s.*

$$T(\mathcal{G}(n, p)) = \min \left\{ \delta(\mathcal{G}(n, p)), \left\lfloor \frac{\bar{d}(\mathcal{G}(n, p))}{2} \right\rfloor \right\}.$$

Theorem 2. *Let $\beta = 2/\log(e/2) \approx 6.51778$. Then*

- (i) *if $p = \frac{\beta(\log n - \log \log n/2) - \omega(1)}{n-1}$, then a.a.s. $\delta(\mathcal{G}(n, p)) \leq \left\lfloor \frac{\bar{d}(\mathcal{G}(n, p))}{2} \right\rfloor$ and so $T(\mathcal{G}(n, p)) = \delta(\mathcal{G}(n, p))$;*
- (ii) *if $p = \frac{\beta(\log n - \log \log n/2) + \omega(1)}{n-1}$, then a.a.s. $\delta(\mathcal{G}(n, p)) > \left\lfloor \frac{\bar{d}(\mathcal{G}(n, p))}{2} \right\rfloor$ and so $T(\mathcal{G}(n, p)) = \left\lfloor \frac{\bar{d}(\mathcal{G}(n, p))}{2} \right\rfloor$.*

Consider the random graph process $G_0, G_1, \dots, G_{\binom{n}{2}}$ defined as follows: for each $m = 0, \dots, \binom{n}{2}$, G_m is a graph with vertex set $[n]$; the graph G_0 has no edges; and, for each $1 \leq m \leq \binom{n}{2}$, the graph G_m is obtained by adding one new edge to G_{m-1} chosen uniformly at random among the edges not present in G_{m-1} . Equivalently, we can choose uniformly at random a permutation $(e_1, \dots, e_{\binom{n}{2}})$ of the edges of the complete graph with vertex set $[n]$, and define each G_m to be the graph on vertex set $[n]$ and edges e_1, \dots, e_m .

Theorem 3. *Let $\beta = 2/\log(e/2) \approx 6.51778$. The following holds in the random graph process $G_0, G_1, \dots, G_{\binom{n}{2}}$.*

- (i) *A.a.s. $T(G_m) = \min \{ \delta(G_m), \lfloor m/(n-1) \rfloor \}$ for every $0 \leq m \leq \binom{n}{2}$.*

- (ii) *Moreover, for any constant $\epsilon > 0$, a.a.s.*

- $\delta(G_m) \leq \lfloor m/(n-1) \rfloor$ for every $0 \leq m \leq \frac{(1-\epsilon)\beta}{2} n \log n$, and
- $\delta(G_m) > \lfloor m/(n-1) \rfloor$ for every $\frac{(1+\epsilon)\beta}{2} n \log n \leq m \leq \binom{n}{2}$.

Remark: We will actually prove a stronger result than Theorem 3 (ii). See Theorem 28 in Section 6.

For any graph G , let $A(G)$ denote the minimum number of subforests of G which cover the whole edge set of G . This number is known as the arboricity of G .

Theorem 4. *Let $\beta = 2/\log(e/2) \approx 6.51778$.*

- (i) *For all $p = \frac{\beta(\log n - \log \log n/2) + \omega(1)}{n-1}$, a.a.s. $A(\mathcal{G}(n, p)) = \left\lceil \frac{\bar{d}(\mathcal{G}(n, p))}{2} \right\rceil$; for all $p = \omega(1/n)$, a.a.s. $A(\mathcal{G}(n, p)) \in \left\{ \left\lceil \frac{\bar{d}(\mathcal{G}(n, p))}{2} \right\rceil, \left\lceil \frac{\bar{d}(\mathcal{G}(n, p))}{2} \right\rceil + 1 \right\}$;*

(ii) For all $p = \Theta(1/n)$, a.a.s. $A(\mathcal{G}(n, p)) = (1 + \Theta(1))pn/2$. Moreover, there exists a $k > 0$ (depending on p), such that a.a.s. $A(\mathcal{G}(n, p)) \in \{k, k + 1\}$.

(iii) If $p = o(1/n)$, then a.a.s. $A(\mathcal{G}(n, p)) \leq 1$.

Remark: (a) Note that Theorem 4 (i) is a simple corollary of Theorem 5 below. Indeed, for most values of $p = \omega(1/n)$, we can even do better than the two-value concentration result stated above and a.a.s. determine the exact value of the arboricity (cf. the remark that follows Theorem 5). (b) It follows from Theorem 4 that, for all $p = \omega(1/n)$, the arboricity of $\mathcal{G}(n, p)$ is asymptotic to $pn/2$, whereas this property fails for $p = O(1/n)$.

Theorem 5. Let $\beta = 2/\log(e/2) \approx 6.51778$. The following holds in the random graph process $G_0, G_1, \dots, G_{\binom{n}{2}}$.

(i) Let m_0 be any function of n such that $m_0/n \rightarrow \infty$ and let $\epsilon > 0$ be any constant. Then, a.a.s. simultaneously for all $m \geq m_0$ such that $\delta(G_m) \leq \bar{d}(G_m)/2$,

$$\left\lceil \frac{m + \phi_1}{n - 1} \right\rceil \leq A(G_m) \leq \left\lceil \frac{m + \phi_2}{n - 1} \right\rceil, \quad (1)$$

where $\phi_1 = n/\exp\left(\frac{(1+\epsilon)2m}{\beta n}\right) = o(n)$ and $\phi_2 = n/\exp\left(\frac{(1-\epsilon)2m}{\beta n}\right) = o(n)$. In particular, a.a.s. $A(G_m) \in \{\lceil \frac{m}{n-1} \rceil, \lceil \frac{m}{n-1} \rceil + 1\}$ for all m in that range.

(ii) Moreover, a.a.s. simultaneously for every m such that $\delta(G_m) \geq \bar{d}(G_m)/2$ we have

$$A(G_m) = \lceil m/(n - 1) \rceil.$$

Remark: Given positive integers a and b , let $R(a, b) = a - b\lfloor a/b \rfloor$ denote the remainder of a divided by b . Then (1) in Theorem 5(a) implies $A(G_m) = \lceil \frac{m}{n-1} \rceil$ for those m such that $0 < R(m, n - 1) \leq n - 1 - \phi_2$ (which is the case for most of the values of m), and $A(G_m) = \lceil \frac{m}{n-1} \rceil + 1$ if $R(m, n - 1) = 0$ or $R(m, n - 1) > n - 1 - \phi_1$. For those few remaining values of m such that $n - 1 - \phi_2 < R(m, n - 1) \leq n - 1 - \phi_1$ we can only say $A(G_m) \in \{\lceil \frac{m}{n-1} \rceil, \lceil \frac{m}{n-1} \rceil + 1\}$.

Corollary 6. Let $m_{A=i}$ denote the minimum m such that $A(G_m)$ becomes i in the random graph process $G_0, G_1, \dots, G_{\binom{n}{2}}$. Let i_0 be any function of n such that $i_0 \rightarrow \infty$ and $\epsilon > 0$ be a constant. Then a.a.s.

(i) for every $i_0 \leq i \leq (1 - \epsilon)\beta \log n/2$,

$$(i - 1)(n - 1) - \phi_2 < m_{A=i} < (i - 1)(n - 1) - \phi_1,$$

where $\phi_1 = n/\exp\left(\frac{2(1+\epsilon)i}{\beta}\right) = o(n)$ and $\phi_2 = n/\exp\left(\frac{2(1-\epsilon)i}{\beta}\right) = o(n)$; and

(ii) for every $(1 + \epsilon)\beta \log n/2 \leq i \leq n/2$,

$$m_{A=i} = (i - 1)(n - 1) + 1.$$

3 Tools

For any graph G and any partition \mathcal{P} of its vertex set, let $m_G(\mathcal{P})$ denote the number of edges in G with ends in distinct parts of \mathcal{P} . When G is implicit from the context, we may drop the subscript.

Theorem 7 (Tutte [26] and Nash-Williams [20]). *Let G be a graph. Then G contains t edge-disjoint spanning trees if and only if, for every partition \mathcal{P} of the vertex set of G such that every class is non-empty,*

$$m_G(\mathcal{P}) \geq t(|\mathcal{P}| - 1). \quad (2)$$

For any graph G and $S \subseteq V(G)$, let $E_G[S]$ denote the set of edges of G with both ends in S . When G is implicit from the context, we may drop the subscript.

Theorem 8 (Nash-Williams [21]). *Let G be a graph. Then the edge set of G can be covered by t forests if and only if, for every nonempty subset S of vertices of G ,*

$$|E_G[S]| \leq t(|S| - 1). \quad (3)$$

Theorem 9 (Chernoff's bounds). *Let X_1, \dots, X_n denote n independent Bernoulli variables. Let $X = \sum_{i=1}^n X_i$ and let $\mu = \mathbf{E}X$. Then for any $0 < t < 1$,*

$$\Pr(X \geq (1+t)\mu) \leq \exp(-t^2\mu/3), \quad \Pr(X \leq (1-t)\mu) \leq \exp(-t^2\mu/2).$$

For the rest of this section, we always let G denote $\mathcal{G}(n, p)$ and let $\delta := \delta(\mathcal{G}(n, p))$ and $\bar{d} := \bar{d}(\mathcal{G}(n, p))$. For any vertex v , let d_v denote the degree of v in G .

Lemma 10. *For any $n \geq 2$ and any function $t(n) < 1$, we have that the probability that $|\bar{d} - pn| \leq tpn$ and $|m - p\binom{n}{2}| \leq tp\binom{n}{2}$ is at least $1 - 2\exp(-At^2pn^2)$ where $A = 1/12$.*

Proof. By the definition of \bar{d} , the events $|\bar{d} - pn| > tpn$ and $|m - p\binom{n}{2}| > tp\binom{n}{2}$ are equivalent. Then, since the number of edges in $\mathcal{G}(n, p)$ is distributed as $\mathbf{Bin}(\binom{n}{2}, p)$, we apply Chernoff's bound in Theorem 9 and obtain

$$\Pr\left(|m - p\binom{n}{2}| > tp\binom{n}{2}\right) \leq 2\exp\left(-\frac{t^2p\binom{n}{2}}{3}\right). \quad \square$$

Lemma 11. *Let $f \geq 0$ be any function of n such that $f \rightarrow \infty$. Then, there exist a constant $C > 0$ such that for every $f/n \leq p \leq 1$ the following holds in $\mathcal{G}(n, p)$ with probability at least $1 - e^{-C(pn)^{1/3}}$. The number of vertices with degrees not in $[\bar{d} - (pn)^{2/3}, \bar{d} + (pn)^{2/3}]$ is at most $n/e^{Cf^{1/3}}$.*

Proof. We have that

$$\Pr(|d_v - \bar{d}| > (pn)^{2/3}) \leq \Pr\left(|\bar{d} - pn| > \frac{(pn)^{2/3}}{2}\right) + \Pr\left(|d_v - pn| > \frac{(pn)^{2/3}}{2}\right).$$

By Lemma 10 with $t = \frac{1}{2} \cdot (pn)^{-1/3}$,

$$\Pr\left(|\bar{d} - pn| \leq \frac{(pn)^{2/3}}{2}\right) \leq 2\exp(-An \cdot (pn)^{1/3}),$$

where A is a positive constant. By Chernoff's inequality in Theorem 9, for a positive constant B ,

$$\Pr\left(|d_v - pn| > \frac{(pn)^{2/3}}{2}\right) \leq 2 \exp\left(-\frac{Bpn}{(pn)^{2/3}}\right) = 2 \exp\left(-B(pn)^{1/3}\right).$$

Thus, there is a positive constant C such that,

$$\Pr(|d_v - \bar{d}| > (pn)^{2/3}) \leq \exp\left(-2C(pn)^{1/3}\right).$$

Thus, by Markov's inequality, the probability that the number of vertices with degree outside $[\bar{d} - (pn)^{2/3}, \bar{d} + (pn)^{2/3}]$ is more than $n \exp(-Cf^{1/3})$ is at most

$$\frac{n \exp\left(-2C(pn)^{1/3}\right)}{n \exp\left(-Cf^{1/3}\right)} \leq \exp\left(-C(pn)^{1/3}\right),$$

since $pn \geq f$. □

Lemma 12. (i) For any $p \leq 0.9 \log n / (n - 1)$, a.a.s. $\delta(\mathcal{G}(n, p)) = 0$.

(ii) For every $0 < \theta < 1$ and every $C > 0$, there exists a $\gamma > 0$, such that for all $p \geq \gamma \log n / (n - 1)$, $\Pr(\delta(\mathcal{G}(n, p)) < \theta p(n - 1)) \leq e^{-C \log n}$.

Proof. Part (i) was proved in [3]. We prove only part (ii). By Theorem 9,

$$\Pr\left(\delta(\mathcal{G}(n, p)) \leq \theta p(n - 1)\right) \leq n \exp\left(-(1 - \theta)^2 p(n - 1)/2\right) \leq n \exp\left(-(1 - \theta)^2 (\gamma/2) \log n\right).$$

Thus, the statement holds by choosing γ sufficiently large so that $(1 - \theta)^2 \gamma/2 - 1 > C$. □

For any sets $S, S' \subseteq [n]$, let $E(S, S')$ be the set of edges in G with one end in S and the other in S' .

Lemma 13. Let $f \geq 0$ be any function of n such that $f \rightarrow \infty$, and $\zeta > 0$ any fixed constant. Then, there exists a constant $C > 0$ such that for every $f/n \leq p \leq 1$ the following holds in $\mathcal{G}(n, p)$ with probability at least $1 - e^{-Cpn^2}$. For every disjoint sets $S, S' \subseteq [n]$ with $|S|, |S'| \geq \zeta n$ we have $|E(S, S')| \geq (\bar{d}/4)|S||S'|/n$.

Proof. The variable $|E(S, S')|$ has distribution $\mathbf{Bin}(|S||S'|, p)$. By Lemma 10 with $t = 1/4$, we have that

$$\Pr\left(\bar{d} \geq \frac{5}{4}pn\right) \leq 2 \exp(-Apn^2)$$

where $A = 1/12$ and $n \geq 2$. By Chernoff's bound in Theorem 9, for a positive constant B ,

$$\Pr\left(\mathbf{Bin}\left(|S||S'|, p\right) < \frac{5p}{16}|S||S'|\right) \leq \exp(-Bp|S||S'|).$$

Hence, the probability that there exist such S and S' is at most

$$\begin{aligned} & 2 \exp\left(-Apn^2\right) + \sum_{s, s' > \zeta n} \binom{n}{s} \binom{n}{s'} \exp(-Bpss') \leq \exp\left(-Apn^2\right) + n^2 \cdot 2^n \cdot 2^n \exp(-B\zeta^2 pn^2) \\ & \leq 2 \exp\left(-Apn^2\right) + \exp(B''n - B\zeta^2 pn^2). \end{aligned}$$

for a positive constant B'' and we are done since $pn \geq f \rightarrow \infty$ as $n \rightarrow \infty$. □

Lemma 14. *Let $f \geq 0$ be any function of n such that $f \rightarrow \infty$, and let $\alpha > 0$ be any fixed constant. Then, there exist constants $\zeta > 0$ and $C > 0$ such that for every $f/n \leq p \leq 1$ the following holds in $\mathcal{G}(n, p)$ with probability at least $1 - Ce^{-(pn)^2}$. For all $s \leq \zeta n$ and every set S of size s , we have that $|E[S]| \leq \alpha pns$.*

Proof. The result is trivial for any set of size $s \leq 2\alpha pn$ since $|E[S]| \leq s^2/2 \leq s(2\alpha pn)/2 = \alpha pns$. Let $\zeta > 0$ be small enough so that $\frac{e\zeta}{2\alpha} < e^{-1/\alpha^2}$. The expected number of sets of size $2\alpha pn \leq s \leq \zeta n$ containing at least αpns edges is at most

$$\begin{aligned} \binom{n}{s} \binom{\binom{s}{2}}{\lceil \alpha pns \rceil} p^{\lceil \alpha pns \rceil} &\leq \left(\frac{en}{s} \left(\frac{es}{2\alpha n} \right)^{\lceil \alpha pns \rceil} \right)^s = \left(\frac{e^2}{2\alpha} \left(\frac{es}{2\alpha n} \right)^{\lceil \alpha pns \rceil - 1} \right)^s \\ &\leq \left(A \left(\frac{e\zeta}{2\alpha} \right)^{\alpha pn} \right)^s < (Ae^{-pn/\alpha})^s, \end{aligned}$$

for some constant $A > 0$ depending only on f and ζ (we used the fact that the exponent $\lceil \alpha pns \rceil - 1 \geq \alpha f - 1$, which eventually becomes positive as $f \rightarrow \infty$).

Summing the expectation above over all $s \geq 2\alpha pn$, we get

$$\sum_{s \geq 2\alpha pn} (Ae^{-pn/\alpha})^s \leq (Ae^{-pn/\alpha})^{2\alpha pn} \frac{1}{1 - Ae^{-f/\alpha}} \leq Ce^{-(pn)^2},$$

for some constant $C > 0$ only depending on f and ζ . \square

Lemma 15. *For every constant $\eta > 0$ there exist positive constants C_1 and C_2 such that the following holds for any function $0 \leq p \leq 1/\sqrt{n}$ and every integer $0 < k \leq (1 - \eta)np$. Let $X \sim \mathbf{Bin}(n, p)$. Then,*

$$\Pr(X \leq k) = C \frac{e^{-pn}}{\sqrt{k}} \left(\frac{epn}{k} \right)^k \quad \text{and} \quad C_1 \leq C \leq C_2.$$

Proof. It follows easily from Stirling's approximation, that there exist two positive constants A_1 and A_2 such that, for every $0 < k < \sqrt{n}$,

$$\frac{A_1}{\sqrt{k}} \left(\frac{en}{k} \right)^k \leq \binom{n}{k} \leq \frac{A_2}{\sqrt{k}} \left(\frac{en}{k} \right)^k.$$

Moreover, there exist positive constants B_1 and B_2 such that, for every $0 \leq p \leq 1/\sqrt{n}$ and every $0 < k < \sqrt{n}$,

$$B_1 e^{-pn} \leq (1 - p)^{n-k} \leq B_2 e^{-pn}.$$

Therefore, there is some $C_1 = \Theta(1)$ not depending on p or k such that

$$\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} = C_1 \frac{e^{-pn}}{\sqrt{k}} \left(\frac{epn}{k} \right)^k,$$

and the lower bound follows immediately since $\Pr(X \leq k) \geq \Pr(X = k)$. For the upper bound, let $f_i = \Pr(X = i) = \binom{n}{i} p^i (1 - p)^{n-i}$, and observe that, for every $i \leq (1 - \eta)np$,

$$\frac{f_{i-1}}{f_i} \leq \frac{i}{(n - i)p} \leq 1 - \frac{\eta}{2},$$

since $p \leq 1/\sqrt{n} \leq \frac{\eta}{2-\eta}$ eventually. Hence, there is a constant $C_2 > 0$ only depending on η such that $\Pr(X \leq k) \leq C_2 \Pr(X = k)$. \square

Lemma 16. *Let Y denote the number of vertices of degree at most k in $\mathcal{G}(n, p)$, where $p < 1$. Then $\mathbf{Var}(Y) \leq (\mathbf{E}Y)^2(p/(1-p) + 1/\mathbf{E}Y)$.*

Proof. Recall that d_v has distribution $\mathbf{Bin}(n-1, p)$. Let $q_{\leq}(r, t)$ denote the probability that a random variable with distribution $\mathbf{Bin}(r, p)$ has value at most t and let $q_{=}(r, t)$ denote the probability that it has value exactly t . Then

$$\begin{aligned} \mathbf{E}(Y^2) &= \sum_{u, v \in V} \mathbf{Pr}(d_v \leq k \text{ and } d_u \leq k) \\ &= \mathbf{E}Y + n(n-1) \left(p \cdot q_{\leq}(n-2, k-1)^2 + (1-p) \cdot q_{\leq}(n-2, k)^2 \right). \end{aligned}$$

This holds because for any distinct vertices $u, v \in V$, the number of neighbours of u in $V \setminus \{v\}$ and the number of neighbours of v in $V \setminus \{u\}$ are independent random variables with distribution $\mathbf{Bin}(n-2, p)$. Clearly, $q_{\leq}(r, t) = q_{\leq}(r, t-1) + q_{=}(r, t)$. And so

$$\begin{aligned} p \cdot q_{\leq}(n-2, k-1)^2 + (1-p) \cdot q_{\leq}(n-2, k)^2 \\ = q_{\leq}(n-2, k-1)^2 + 2(1-p) \cdot q_{=}(n-2, k)q_{\leq}(n-2, k-1) + (1-p) \cdot q_{=}(n-2, k)^2. \end{aligned} \quad (4)$$

Moreover

$$q_{\leq}(n-1, k)^2 = \left(q_{\leq}(n-2, k-1) + (1-p) \cdot q_{=}(n-2, k) \right)^2$$

Thus,

$$\begin{aligned} \mathbf{E}(Y^2) &= \mathbf{E}Y + n(n-1) \left(q_{\leq}(n-1, k)^2 + (1-p)q_{=}(n-2, k)^2(1 - (1-p)) \right) \\ &\leq \mathbf{E}Y + n^2 q_{\leq}(n-1, k)^2 \left(1 + \frac{p}{1-p} \right) = \mathbf{E}(Y)^2 \left(1 + \frac{p}{1-p} + \frac{1}{\mathbf{E}(Y)} \right) \end{aligned}$$

□

Next, we introduce two lemmas that estimate $\delta(\mathcal{G}(n, p))$. These lemmas are used when we require more precise probability bounds given by the Chernoff's bounds in Theorem 9. The first lemma estimates this parameter for $0.9 \log n/(n-1) \leq p \leq \gamma \log n/(n-1)$ where $\gamma \geq 0.9$ is a constant.

Lemma 17. *Let $\gamma \geq 0.9$ and $0 < \eta < 1$ be constants and let $\alpha = \alpha(n)$ such that $0 < \alpha \leq 1-\eta$. Then there exists a constant $C > 0$ such that, for every $0.9 \log n/(n-1) \leq p \leq \gamma \log n/(n-1)$,*

$$\begin{aligned} (i) \quad & \mathbf{Pr}\left(\delta \leq \alpha p(n-1)\right) \leq C \exp \left(\log n - p(n-1) \left(1 - \alpha \log \left(\frac{e}{\alpha} \right) \right) - \frac{1}{2} \log \log n \right) \text{ and} \\ (ii) \quad & \mathbf{Pr}\left(\delta > \alpha p(n-1)\right) \leq C \left(\frac{\log n}{n} + \exp \left(p(n-1) \left(1 - \alpha \log \left(\frac{e}{\alpha} \right) \right) - \log n + \frac{1}{2} \log \log n \right) \right). \end{aligned}$$

Proof. Given an arbitrary vertex v , let $d_v \sim \mathbf{Bin}(n-1, p)$ be the degree of v . By Lemma 15, there exists a function $C' \in [C_1, C_2]$, where C_1, C_2 are positive constants that depend only on α and γ such that

$$\mathbf{Pr}(d_v \leq \alpha p(n-1)) = C' \exp \left(-p(n-1) \left(1 - \alpha \log \left(\frac{e}{\alpha} \right) \right) - \frac{1}{2} \log \log n \right). \quad (5)$$

Set C to be a constant larger than $C_2 + 2\gamma + 1/C_1 \geq C' + 2\gamma + 1/C'$. From (5), the expected number of vertices with degree at most $\alpha p(n-1)$ is

$$C' \exp \left(\log n - p(n-1) \left(1 - \alpha \log \left(\frac{e}{\alpha} \right) \right) - \frac{1}{2} \log \log n \right),$$

which implies (i), since $C \geq C'$. Finally, the proof of (ii) follows from Lemma 16 and Chebyshev's inequality, since $p/(1-p) \leq 2\gamma \log n/n$ and $C \geq 2\gamma + 1/C'$. \square

The following lemma gives a fairly precise estimate of the probability that $\delta(\mathcal{G}(n, p))$ shifts slightly from $pn/2$, when p is very close to $\beta \log n/(n-1)$. It is normally applied by choosing ϵ so that ϵpn is negligible comparing to the other terms in (6) and (7).

Lemma 18. *Let $\gamma > \beta$ be a constant. Let $p = \frac{\beta \log n + f(n)}{n-1} \leq \frac{\gamma \log n}{n}$. Let v be an arbitrary vertex and denote by d_v the degree of v . Let $\epsilon = \epsilon(n)$ be such that $|\epsilon| < 1$. Then there exists a positive constant C such that*

$$\Pr \left(\delta < \frac{(1-\epsilon)}{2} pn \right) \leq \exp \left(-\frac{f}{\beta} - \frac{1}{2} \log \log n + C(|\epsilon|pn + 1) \right) \quad (6)$$

and

$$\Pr \left(d_v \leq \frac{(1+\epsilon)}{2} pn \right) \geq \exp \left(-\log n - \frac{f}{\beta} - \frac{1}{2} \log \log n - C(|\epsilon|pn + 1) \right).$$

and there is a positive constant D such that

$$\Pr \left(\delta > \frac{(1+\epsilon)}{2} pn \right) \leq D \left(\frac{\log n}{n} + \exp \left(+\frac{f}{\beta} + \frac{1}{2} \log \log n + C(|\epsilon|pn + 1) \right) \right). \quad (7)$$

Proof. Given an arbitrary vertex v , let $d_v \sim \mathbf{Bin}(n-1, p)$ be the degree of v . By Lemma 15, there exists a positive constant C_1 such that

$$\begin{aligned} \Pr \left(d_v \leq \frac{(1-\epsilon)}{2} pn \right) &\leq C_1 \left(\exp \left(-pn + \frac{(1-\epsilon)}{2} pn \log \left(\frac{2e}{1-\epsilon} \right) - \frac{1}{2} \log \log n \right) \right) \\ &\leq \exp \left(-pn \left(1 - \frac{1}{2} \log(2e) \right) + C_2(|\epsilon|pn) - \frac{1}{2} \log \log n \right), \end{aligned}$$

since $p \leq \gamma \log n/(n-1)$ and for a positive constant C_2 . Using $\beta \left(1 - \frac{\log(2e)}{2} \right) = 1$ and union bound, we have that

$$\Pr(\delta \leq \alpha(1-\epsilon)pn) \leq \exp \left(\frac{-f}{\beta} - \frac{1}{2} \log \log n + C_3(pn|\epsilon| + 1) \right),$$

for another positive constant C_3 . Thus, we proved (6) (by choosing $C > C_3$). Now we prove (7). Using Lemma 15 again and using similar calculation as before, for a positive constant C_4 ,

$$\Pr \left(d_v \leq \frac{1+\epsilon}{2} pn \right) \geq \exp \left(-\log n - \frac{f}{\beta} - \frac{1}{2} \log \log n - C_4(pn|\epsilon| + 1) \right),$$

and (7) follows from the above together with Lemma 16 and Chebyshev's inequality (by choosing $D = 2\gamma$ and $C > C_4$). \square

Lemma 19. *Let $\gamma > 1$ be a constant. There exist positive constants ϵ and C such that, for $p \geq \gamma \log n / (n - 1)$, we have that $\delta > \epsilon pn$ with probability at least $1 - e^{-C \log n}$.*

Proof. By Lemma 12 (ii), there is a $\rho > 1$ such that with probability at least $1 - e^{-\log n}$, for all $p \geq \rho \log n / (n - 1)$, $\delta > (1/2)pn$. Thus, for $\gamma \geq \rho$, the lemma follows immediately by choosing any $\epsilon \leq 1/2$ and any $C \leq 1$.

So suppose $1 < \gamma < \rho$. By Lemma 17, there is a positive constant C' so that

$$\Pr(\delta < \epsilon pn) \leq \exp \left(-\gamma \log n \left(1 - \epsilon \log \left(\frac{\epsilon}{\epsilon} \right) \right) + \log n + C' \right).$$

Since $\gamma > 1$, we can choose $0 < \epsilon \leq 1/2$ small enough so that $\gamma(1 - \epsilon \log(\frac{\epsilon}{\epsilon})) > 1$. Then there is $C'' > 0$ such that the above probability is at most $e^{-C'' \log n}$. The lemma follows by choosing $C = \min\{1, C''\}$. \square

Lemma 20. *For any constants $1 < \gamma_1 < \gamma_2$ and $\eta(n) = o(1)$, there exist $\epsilon(n) = o(1)$ and a constant $C > 0$ such that for any functions $\gamma_1 \log n / (n - 1) \leq p \leq p' \leq \gamma_2 \log n / (n - 1)$ such that $|p/p' - 1| \leq \eta$ the following holds with probability at least $1 - e^{-C \log n}$:*

$$\left| \frac{\delta(\mathcal{G}(n, p))}{\delta(\mathcal{G}(n, p'))} - 1 \right| \leq \epsilon.$$

Proof. Apply Lemma 19 with $\gamma = \gamma_1$ and let ϵ' be the positive constant ϵ given by Lemma 19. Then, for $\gamma := \epsilon' \gamma_1$

$$\Pr(\delta(\mathcal{G}(n, p')) > \gamma \log n) \geq 1 - e^{-A \log n},$$

for a positive constant A . Let $\theta = \gamma \epsilon$. Using the usual coupling for $\mathcal{G}(n, p)$ and $\mathcal{G}(n, p')$, we get that the number d'_v of edges incident to a vertex v present in $\mathcal{G}(n, p')$ and not in $\mathcal{G}(n, p)$ is dominated by $\text{Bin}(n, q)$ where $q = (p' - p)/(1 - p)$. Let ζ be such that $(1 + \zeta)qn = \theta \log n$. By Chernoff's bound and union bound, there is a vertex v with $d'_v \geq \theta \log n$ is at most

$$\begin{aligned} n \Pr(d'_v \geq \theta \log n) &\leq n \exp(qn(\zeta - (1 + \zeta) \log(1 + \zeta))) \\ &= n \exp\left(\theta \log n - qn - \theta \log n \log\left(\frac{\theta \log n}{qn}\right)\right) \\ &\leq \exp\left(\log n + \theta \log n - \theta \log n \log\left(\frac{\theta}{\eta}\right)\right), \end{aligned}$$

so we can choose $\epsilon = \theta/\gamma$ as $\frac{3}{\gamma \log(1/\eta)}$ which goes to 0 since η goes to 0. Then $\theta \log(\theta/\eta) = \theta \log \theta + \theta \log(1/\eta) > 2 + \theta$ eventually and so with probability at least $1 - \exp(-\log n) - \exp(-A \log n)$,

$$\delta(\mathcal{G}(n, p')) - \delta(\mathcal{G}(n, p)) \leq \theta \log n \leq \epsilon \gamma \log n \leq \epsilon \delta(\mathcal{G}(n, p')).$$

\square

Lemma 21. *For any constant $\epsilon > 0$, there exist constants $\gamma_1 > 1$ and $C > 0$ such that, for any $\frac{0.9 \log n}{n-1} \leq p \leq \frac{\gamma_1 \log n}{n-1}$, we have that with probability at least $1 - e^{-C \log n}$, $\delta(\mathcal{G}(n, p)) \leq \epsilon \bar{d}(\mathcal{G}(n, p))$.*

Proof. We may assume $\epsilon < 1$. Let $\alpha = 0.8/\gamma_1$. We have that

$$\Pr(\delta \geq \epsilon \bar{d}) \leq \Pr(\bar{d} \leq 0.8 \log n) + \Pr(\delta \geq \epsilon 0.8 \log n)$$

By Lemma 10, for any $p \geq 0.9 \log n / (n - 1)$, for a positive constant A ,

$$\Pr(\bar{d} \leq 0.8 \log n) \leq 2 \exp(-A p n^2). \quad (8)$$

Let $p = \gamma_1 \log n / (n - 1)$. By Lemma 17, in $\mathcal{G}(n, p)$,

$$\begin{aligned} \Pr(\delta \geq \epsilon 0.8 \log n) &\leq \Pr(\delta > \epsilon \alpha p (n - 1)) \\ &\leq C' \left(\log n / n + \exp \left(\gamma_1 \log n \left(1 - \epsilon \alpha \log \left(\frac{e}{\epsilon \alpha} \right) \right) - \log n + \frac{1}{2} \log \log n + D \right) \right), \end{aligned}$$

For any $\epsilon \in (0, 1)$, we have that $(1 - \epsilon \log(\frac{e}{\epsilon})) < 1$. So by choosing $\gamma_1 > 1$ small enough we have that $B := \gamma_1 (1 - \epsilon \alpha \log(\frac{e}{\epsilon \alpha})) < 1$ and the above probability is at most

$$C' \left(\log n / n + \exp \left(- (1 - B) \log n + D + \frac{1}{2} \log \log n \right) \right). \quad (9)$$

By choosing C sufficiently small, the probabilities in (8) and in (9) are both smaller than $(1/2)e^{-C \log n}$. By monotonicity, for all $p \leq \gamma_1 \log n / (n - 1)$, $\Pr(\delta(\mathcal{G}(n, p)) \geq \epsilon 0.8 \log n)$ is at most the probability in (9). Thus, with probability at least $1 - e^{-C \log n}$, for all $0.9 \log n / (n - 1) \leq p \leq \gamma_1 \log n / (n - 1)$, we have $\delta(\mathcal{G}(n, p)) \leq \epsilon \bar{d}(\mathcal{G}(n, p))$. \square

For $\epsilon > 0$, we say that a vertex is ϵ -light if its degree is at most $\delta + \epsilon d$.

Lemma 22. *Suppose $0.9 \log n / (n - 1) \leq p \leq \gamma \log n / (n - 1)$ for some constant $\gamma \geq 0.9$. Then there exist constants $\epsilon > 0$ and $C > 0$ such that the following holds in $\mathcal{G}(n, p)$ with probability at least $1 - e^{-C \log n}$. There is no pair of adjacent ϵ -light vertices and no two ϵ -light vertices have a common neighbour.*

Proof. Let $x = p(n - 1) / \log n$. For each $x \in [0.9, \gamma]$, define $\alpha = \alpha(x)$ to be the only solution in $(0, 1)$ of

$$x(1 - \alpha \log(e/\alpha)) = 0.8. \quad (10)$$

It is straightforward to verify that $\alpha \in (0, 1)$ is well defined and strictly increasing with respect to $x \in [0.9, \gamma]$. Consider the constant $\hat{\epsilon} = 0.1/(\gamma - 0.8)$, and define $\hat{\alpha} = (1 + \hat{\epsilon})\alpha$. Recall that both α and $\hat{\alpha}$ are functions of $x = p(n - 1) / \log n$. Then, using (10) and the fact that $\hat{\epsilon} \leq 0.1/(x - 0.8)$, we obtain

$$x(1 - \hat{\alpha} \log(e/\hat{\alpha})) > x(1 - \hat{\alpha} \log(e/\alpha)) = x - (1 + \hat{\epsilon})(x - 0.8) \geq 0.7. \quad (11)$$

From (10) and by Lemma 17 (ii), we can bound

$$\Pr(\delta > \alpha p (n - 1)) \leq D e^{-0.19 \log n}, \quad (12)$$

for a constant $D > 0$ not depending on p . Assume for the rest of the argument that D is sufficiently large. Let S be the set of vertices of degree at most $\hat{\alpha}p(n-1)$. By (5) in the proof of Lemma 17 and (11), the probability that a vertex v belongs to S is

$$\Pr(v \in S) = \Pr(d_v \leq \hat{\alpha}p(n-1)) \leq De^{-0.7 \log n}. \quad (13)$$

We can upper-bound the probability that a pair of vertices u and v are adjacent and belong to S , by

$$p\Pr(d_u \leq \hat{\alpha}p(n-1))\Pr(d_v \leq \hat{\alpha}p(n-1)) = p(\Pr(v \in S))^2.$$

Multiplying this by the number of possible pairs and using (13), we get that the probability that S contains some adjacent pair of vertices is at most

$$\binom{n}{2}p(\Pr(v \in S))^2 \leq D\gamma \log n e^{-0.4 \log n}.$$

By a similar argument, the probability that S contains a pair of vertices with a common neighbour is at most

$$\binom{n}{2}(n-2)p^2(\Pr(v \in S))^2 \leq D\gamma^2 \log^2 n e^{-0.4 \log n}.$$

Finally, we define $\epsilon = \alpha(0.9)\hat{\epsilon}/2$. Recall that α is increasing in $[0.9, \gamma]$, and then $\epsilon \leq \alpha\hat{\epsilon}/2$. It follows from Lemma 10 that $\bar{d} = \bar{d}(\mathcal{G}(n, p))$ is at most $2p(n-1)$ with probability at least $1 - De^{-\log n}$, assuming that D is large enough. If this event and the one in (12) hold together, then

$$\delta + \epsilon\bar{d} \leq (\alpha + 2\epsilon)p(n-1) \leq \hat{\alpha}p(n-1),$$

and therefore all ϵ -light vertices are contained in S . Putting everything together, the statement holds with probability at least $1 - e^{-C \log n}$, for some small enough constant $C > 0$. \square

The following lemma, which will be used in the proofs in Section 6, follows with almost exactly the same proof as the previous one. For coupling two random graphs $G_1 \subseteq G_2$ such that $G_1 \sim \mathcal{G}(n, p_1)$, $G_2 \sim \mathcal{G}(n, p_2)$ for $p_1 \leq p_2$, we refer readers to [16].

Lemma 23. *Suppose $0.9 \log n/(n-1) \leq p \leq p' \leq \gamma \log n/(n-1)$ for some constant $\gamma \geq 0.9$. Let $G_1 \subseteq G_2$ where $G_1 \sim \mathcal{G}(n, p)$ and $G_2 \sim \mathcal{G}(n, p')$. Then there exist constants $\epsilon > 0$ and $C > 0$ such that the following holds in $\mathcal{G}(n, p)$ and $\mathcal{G}(n, p')$ with probability at least $1 - e^{-C \log n}$. Let S be the set of ϵ -light vertices in G_1 . Then in G_2 , there is no internal edges inside S and no two vertices in S adjacent to a common vertex.*

For any $S \subseteq [n]$, let \bar{S} denote $[n] \setminus S$.

Lemma 24. *Let $\gamma > 1$ be a fixed constant. There exists a constant $C > 0$ such that for any $p = p(n) \geq \gamma \log n/(n-1)$, the following holds in $\mathcal{G}(n, p)$ with probability at least $1 - e^{-C \log n}$. For every $S \subsetneq [n]$ with $2 \leq |S| \leq n-2$, $|E(S, \bar{S})| \geq 1.5\delta$.*

Proof. Without loss of generality, we may assume that $|S| \leq |\bar{S}|$. Since $p \geq \gamma \log n/(n-1)$ for some $\gamma > 1$, by Lemma 19, there exist constants $\epsilon > 0$ and $C_1 > 0$ such that with probability at least $1 - e^{-C_1 \log n}$, $\delta = \delta(\mathcal{G}(n, p)) \geq \epsilon pn$. Let $\alpha = \epsilon/8$. Then by Lemma 14, there exist

constants $\zeta > 0$ and $C_2 > 0$ such that with probability at least $1 - e^{-C_2 \log n}$, for all sets S with size at most ζn , $|E(S, S)| \leq \alpha p n |S|$. Then, with probability at least $1 - e^{-C_1 \log n} - e^{-C_2 \log n}$, for all these S , $|E(S, \bar{S})| \geq \delta |S| - 2\alpha p n |S| \geq (3/4)\delta |S| \geq 1.5\delta$, as $|S| \geq 2$. Now by Lemmas 13 and 10, there exists another constant $C_3 > 0$ such that with probability at least $1 - e^{-C_3 \log n}$, for all sets S with size at least ζn , $|E(S, \bar{S})| \geq (\bar{d}/4)\zeta^2 n$, where $\bar{d} = \bar{d}(\mathcal{G}(n, p)) \sim np$. Clearly, $(\bar{d}/4)\zeta^2 n \geq 1.5\delta$ with probability at least $1 - e^{-C_4 \log n}$ for some $C_4 > 0$. The lemma follows by choosing $C < \min\{C_i : 1 \leq i \leq 4\}$. \square

Lemma 25. *Suppose that $p \leq \gamma \log n / (n - 1)$ for some constant $\gamma > 0$. Then there exist positive constants C and K such that with probability at least $1 - e^{-C \log n}$, the maximum degree of $\mathcal{G}(n, p)$ is at most $K \log n$.*

Proof. Let Δ denote the maximum degree in G and d_v denote the degree of v in G for any vertex v . By union bound and Chernoff's bound, for any $K > 0$,

$$\Pr(\Delta \geq K \log n) \leq n \Pr(d_v \geq K \log n) \leq \exp \left(\frac{(K \log n - pn)^2}{pn} - \log n \right)$$

since $pn \leq \gamma \log n$ it suffices to choose α large enough so that $(K - \gamma)^2 / \gamma - 1 > 0$. \square

4 Proof of Theorem 1

We first give two deterministic lemmas:

Lemma 26. *Let $G = G_n$ be a graph on vertex set $[n]$. Let $\delta := \delta(G)$ and let $\bar{d} := \bar{d}(G)$. Suppose that $\bar{d} \rightarrow \infty$ with $n \rightarrow \infty$ and that there exist constants $\epsilon, \zeta, \eta > 0$ such that the following hold, for all sufficiently large n .*

- (a) *The minimum degree δ is at most $(\epsilon/4)\bar{d}$; there is no pair of adjacent ϵ -light vertices; and all vertices of G have at most one ϵ -light neighbour.*
- (b) *All sets of size $s < \zeta n$ contain at most $(\epsilon/4)\bar{d}s$ internal edges.*
- (c) *For all disjoint $S, S' \subseteq [n]$ with $|S| \geq |S'| \geq \zeta n$, we have that $|E(S, S')| \geq \eta \bar{d} n$.*

Then eventually $T(G) = \delta$.

Proof. We will show that every partition of the vertices of G satisfies (2) with $t = \delta$, and thus G has δ edge-disjoint spanning trees by Theorem 7.

Let \mathcal{P} be a partition of $V(G)$. Parts of size one are denoted *singletons*, and singletons consisting of one ϵ -light vertex are called *ϵ -light singletons*. We may assume that

$$\text{every part with size at least 2 has one vertex that is not } \epsilon\text{-light.} \quad (14)$$

This is because, given a part of size at least 2 and with only ϵ -light vertices, we can refine the partition by turning each vertex in this part into a singleton, and this increases the number of parts without increasing the number of edges with ends in distinct parts by Condition (a).

Let \mathcal{K}_1 denote the set of ϵ -light singletons, let \mathcal{K}_2 denote the set of parts of size between 2 and ζn together with the singletons that are not ϵ -light, and let \mathcal{K}_3 denote the set of other parts. For $i = 1, 2, 3$, let $k_i = |\mathcal{K}_i|$. Then, $|\mathcal{P}| = k_1 + k_2 + k_3$.

By Condition (a), no ϵ -light vertices are adjacent. Thus, the number of edges incident with a vertex in \mathcal{K}_1 is at least δk_1 . Suppose \mathcal{K}_2 is non-empty and suppose S is a part in \mathcal{K}_2 , and let r be the number of vertices in S that are not ϵ -light. By the assumption in (14) and the definition of \mathcal{K}_2 , we must have $1 \leq r \leq \zeta n$. The number of edges between these r vertices is at most $(\epsilon/4)\bar{d}r$ by Condition (b). Since these vertices are not ϵ -light, each of them has degree at least $\delta + \epsilon\bar{d}$. By Condition (a), each of these vertices is adjacent to at most one ϵ -light vertex. Thus,

$$|E(S, \bar{S} \setminus \mathcal{K}_1)| \geq r(\delta + \epsilon\bar{d} - 1) - 2(\epsilon/4)\bar{d}r \geq \delta + (\epsilon/4)\bar{d},$$

where the term -1 in the first inequality accounts for a possible ϵ -light neighbour of each one of these r vertices and we use the fact that $d \rightarrow \infty$. Thus, the number of edges in the partition \mathcal{P} is at least

$$m(\mathcal{P}) \geq \delta k_1 + \frac{\delta + (\epsilon/4)\bar{d}}{2} k_2 \geq \delta(k_1 + k_2), \quad (15)$$

as $\delta \leq (\epsilon/4)\bar{d}$ by Condition (a). If $k_3 \leq 1$, this already shows that $m(\mathcal{P}) \geq \delta(|\mathcal{P}| - 1)$. Otherwise, we have $2 \leq k_3 \leq 1/\zeta$, and the number of edges between any two parts of \mathcal{K}_3 is at least $\eta\bar{d}n$ by Condition (c). We can add these additional edges to (15) and obtain

$$m(\mathcal{P}) \geq \delta(k_1 + k_2) + \eta\bar{d}n \geq \delta(k_1 + k_2) + (\epsilon/4)\bar{d}k_3 \geq \delta|\mathcal{P}|, \quad (16)$$

since eventually $\eta n \geq (\epsilon/4)/\zeta \geq (\epsilon/4)k_3$, and $\delta \leq (\epsilon/4)\bar{d}$. \square

Lemma 27. *Let $G = G_n$ be a graph on $[n]$. Let $\delta := \delta(G)$ and $\bar{d} := \bar{d}(G)$, and suppose that $\delta = \Omega(\bar{d})$ and $\bar{d} = \omega(1)$ as $n \rightarrow \infty$. Let $t = \min\{\delta, \bar{d}/2\}$. Suppose moreover that there exist constants $\epsilon, \eta, \zeta > 0$ such that the following hold, for sufficiently large n .*

- (a') *Either we have that $\delta > \frac{(1+\epsilon)\bar{d}}{2}$; or there are no adjacent ϵ -light vertices and each vertex of G is adjacent to at most one ϵ -light vertex.*
- (b') *For all $S \subseteq V(G)$, with $|S| \geq \zeta n$, we have that $d(S) \geq \bar{d}(1 - o(1))$, where $d(S)$ denotes the sum of degrees of vertices in S divided by $|S|$.*
- (c') *For all disjoint $S, S' \subseteq V(G)$ with $|S| \geq |S'| \geq \zeta n$, we have that $|E(S, S')| \geq \eta\bar{d}|S||S'|/n$.*
- (d') *For all $\emptyset \subsetneq S \subsetneq V(G)$, we have that $|E(S, \bar{S})| \geq t$.*
- (e') *All sets of size $s < \zeta n$ contain at most $\min\{\epsilon ts, ts/4\}$ internal edges.*

Then eventually $T(G) = t$.

Proof. We will show that every partition of the vertices of G satisfies (2), and thus G has t edge-disjoint spanning trees by Theorem 7.

We say that a set $S \subseteq V$ is *large* if $|S| \geq \zeta n$. We say that a partition of V is *simple* if each class either is large or a singleton (that is, it consists of a single vertex). Recall that $m(\mathcal{P})$ denotes the number of edges with ends in distinct parts of \mathcal{P} .

Claim 1. If \mathcal{P} is a simple partition, then $m(\mathcal{P}) \geq t(|\mathcal{P}| - 1)$.

Assume Claim 1 holds, and suppose for a contradiction that there is a partition \mathcal{P} of V such that $m(\mathcal{P}) < t(|\mathcal{P}| - 1)$. By Claim 1, \mathcal{P} is not a simple partition. Given a set S and a vertex $v \in S$, let $d_S(v)$ denote the number of neighbours of v inside S . Since \mathcal{P} is not simple, we can find a non-large part S of \mathcal{P} with at least 2 vertices. By Condition (e'), S must contain one vertex w with

$$d_S(w) \leq \frac{2|E[S]|}{|S|} \leq \frac{2t|S|}{4|S|} = t/2. \quad (17)$$

Moreover, condition (d') implies that

$$m(\mathcal{P}) \geq (t/2)(|\mathcal{P}| - 1). \quad (18)$$

Let \mathcal{P}' be obtained from \mathcal{P} by turning w into a singleton. We have $|\mathcal{P}'| = |\mathcal{P}| + 1$ and $m(\mathcal{P}') = m(\mathcal{P}) + d_S(w)$. Combining this facts together with (17) and (18), we obtain

$$\frac{m(\mathcal{P}')}{|\mathcal{P}'| - 1} = \frac{m(\mathcal{P}) + d_S(w)}{|\mathcal{P}|} \leq \frac{m(\mathcal{P}) + t/2}{|\mathcal{P}|} \leq \frac{m(\mathcal{P}) + \frac{m(\mathcal{P})}{|\mathcal{P}| - 1}}{|\mathcal{P}|} = \frac{m(\mathcal{P})}{|\mathcal{P}| - 1}. \quad (19)$$

Repeat this procedure of turning vertices into singletons until no parts of size between 2 and ζn remain, and therefore obtain a simple partition \mathcal{P}'' . Since (19) holds in each iteration, we have

$$\frac{m(\mathcal{P}'')}{|\mathcal{P}''| - 1} \leq \frac{m(\mathcal{P})}{|\mathcal{P}| - 1} < t,$$

which contradicts Claim 1.

To complete the argument, we proceed to prove Claim 1. Let \mathcal{P} be a simple partition. If all parts of \mathcal{P} are singletons, then we have $m(\mathcal{P}) = \frac{\bar{d}}{2}(n - 1) = \frac{\bar{d}}{2}(|\mathcal{P}| - 1)$. Suppose otherwise there is at least one large part. Since \mathcal{P} is simple, each large part has at least ζn vertices and so there are at most $\ell := 1/\zeta = O(1)$ large partitions. Let k be the number of singletons in \mathcal{P} . Note that $k \leq (1 - \zeta)n$ since any large part has at least ζn vertices.

Suppose first that $\zeta n \leq k \leq (1 - \zeta)n$. Then the average degree of the singletons is at least $\bar{d}(1 - o(1))$ by Condition (b'). Since there is at least one large part, the number of edges between the k singletons and this large parts is at least $\eta\zeta\bar{d}k$ by Condition (c'). Hence, $m(\mathcal{P})$ is at least the number of edges incident with a singleton, which is at least

$$\frac{\bar{d}(1 - o(1))k + \eta\zeta\bar{d}k}{2} \geq \frac{(1 + \eta\zeta/2)k}{k + \ell - 1}(\bar{d}/2)(k + \ell - 1).$$

this satisfies Equation (2) with $\bar{d}/2 \geq t$ for large enough n , since $k \geq \zeta n$ and $\ell = O(1)$.

Suppose otherwise that $0 \leq k \leq \zeta n$. By Condition (a'), we have that either $\delta > \frac{(1+\epsilon)\bar{d}}{2}$; or there are no adjacent ϵ -light vertices and each vertex is adjacent to at most one ϵ -light vertex. The number of edges between singletons is at most ϵtk by Condition (e'). In the first case where $\delta > \frac{(1+\epsilon)\bar{d}}{2}$, the total number of edges incident to the singletons is at least

$$\frac{(1 + \epsilon)\bar{d}}{2}k - \epsilon tk \geq (1 + \epsilon)tk - \epsilon tk \geq tk. \quad (20)$$

Now we consider the second case. Recall that a vertex is ϵ -light if it has degree at most $\delta + \epsilon\bar{d}$. Suppose that there are no adjacent ϵ -light vertices and each vertex is adjacent to at

most one ϵ -light vertex. Let K_1 denote the set of singletons that are ϵ -light and K_2 the set of other singletons (singletons that are not ϵ -light). Let $k_i = |K_i|$ for $i = 1, 2$, so $k = k_1 + k_2$ (possibly $k_1, k_2 = 0$). Since there are no adjacent ϵ -light vertices, $|E(K_1, \overline{K_1})| \geq \delta k_1$. Since no two ϵ -light vertices have a common neighbour, we have $d_{[n] \setminus K_1}(v) \geq \delta + \epsilon \bar{d} - 1$, for every $v \in K_2$. Moreover, Condition (e') guarantees that there are at most $\epsilon t k_2 \leq \epsilon \bar{d} k_2 / 2$ edges inside K_2 , and therefore $|E(K_2, \overline{K_2} \setminus K_1)| \geq (\delta + \epsilon \bar{d} - 1)k_2 - \epsilon \bar{d} k_2 / 2$. Thus, the total number of edges incident with singletons is at least

$$\delta k_1 + (\delta + \epsilon \bar{d} - 1)k_2 - \epsilon \bar{d} k_2 / 2 \geq \delta k \geq tk, \quad (21)$$

eventually as $\bar{d} = \omega(1)$ by our assumption. Thus, we have proved that in both cases, the number of edges incident with singletons is at least tk . If the number of large parts is exactly 1, then (2) holds as $|\mathcal{P}| = k + 1$ and $m(\mathcal{P}) \geq tk$ by (20) and (21). Otherwise, if there are at least two large parts, the number of edges between any two of them is at least $\eta \zeta^2 \bar{d} n$ by Condition (c'). Thus, for large enough n ,

$$m(\mathcal{P}) \geq tk + \eta \zeta^2 \bar{d} n \geq t \left(k + \frac{\eta \zeta^2 \bar{d} n}{t} \right) \geq t(k + \ell - 1),$$

since $t \leq \bar{d}/2$ and $\ell = O(1)$. \square

We proceed to prove Theorem 1, as a consequence of Lemmas 26 and 27. For the rest of the argument, let $\delta := \delta(\mathcal{G}(n, p))$ and let $\bar{d} := \bar{d}(\mathcal{G}(n, p))$. We split the argument into cases depending on the range of p .

First observe that by Lemma 12 (i) we can assume that $p \geq 0.9 \log n / (n - 1)$, since for $p \leq 0.9 \log n / (n - 1)$ $\mathcal{G}(n, p)$ is a.a.s. disconnected and the minimum degree is zero, the statement of Theorem 1 holds trivially.

Let γ_2 be a large enough constant so that for $p \geq \gamma_2 \log n / (n - 1)$ we have $\delta > \frac{(3/4)\bar{d}}{2}$ a.a.s. (see Lemma 12 (ii) and Lemma 10). Let $\epsilon < 3/4$ be the constant given by Lemma 22 with $\gamma = \gamma_2$. Let $1 < \gamma_1 < \gamma_2$ be the constant given by Lemma 21 with $\epsilon/4$.

For $0.9 \log n / (n - 1) \leq p \leq \gamma_1 \log n / (n - 1)$, we only need to show that $\mathcal{G}(n, p)$ a.a.s. satisfies the hypothesis of Lemma 26. First, we note from Lemma 10 that $\bar{d} \sim pn \rightarrow \infty$. Condition (a) holds by our choice of γ_1 . Condition (b) follows from Lemma 14 with any $\alpha < \epsilon/4$, since $\bar{d} \sim pn$. Condition (c) is a consequence from Lemma 13 replacing η by η/ζ^2 .

Finally, we show that $\mathcal{G}(n, p)$ a.a.s. satisfies the conditions in Lemma 27 for the range $p \geq \gamma_1 \log n / (n - 1)$. First note that $\delta = \Omega(\bar{d})$ by Lemma 19. Condition (a') is satisfied for $p \geq \gamma_2 \log n / (n - 1)$ since $\delta > \frac{(1+\epsilon)\bar{d}}{2}$ a.a.s. and for $\gamma_1 \log n / (n - 1) \leq p \leq \gamma_2 \log n / (n - 1)$ since no ϵ -light vertices are adjacent nor have a common neighbours a.a.s. by our choice of γ_2 . Condition (b') holds a.a.s. by Lemma 11. Condition (c') holds a.a.s. by Lemma 13. Condition (d') holds a.a.s. by Lemma 24. For condition (e'), note that $\epsilon t / \bar{d}$ is bounded away from 0 since $\delta = \Omega(\bar{d})$. Therefore the condition follows from Lemma 14 with $\alpha = \epsilon t / \bar{d}$.

5 Proof of Theorem 2

Let ω_n be a positive-valued function of n that goes to infinity arbitrarily slowly as $n \rightarrow \infty$. Let $p = \frac{\beta \log n + f(n)}{n-1}$, and assume $f(n) \geq -\beta \log \log n / 2 + \omega_n$. The number of edges in $G \sim \mathcal{G}(n, p)$ is

a binomial random variable distributed as $\mathbf{Bin}(\binom{n}{2}, p)$. If $p < 0.9 \log n/n$, then by Lemma 12 (i), a.a.s. $\delta(G) = 0$ and thus a.a.s. $\delta(G) \leq \bar{d}(G)$. Assuming $p \geq 0.9 \log n/n$. By Chebyshev's inequality, a.a.s. $|m(G) - pn(n-1)/2| \leq \omega_n \sqrt{pn}$, i.e. a.a.s. $|\bar{d}/2 - pn/2| \leq \omega_n \sqrt{p}$, where $\bar{d} = \bar{d}(G)$. By Lemma 12 (ii), there is a constant $\gamma > 0$, such that for all $p \geq \gamma \log n/n$, a.a.s. $\delta(G) \geq (3/4)pn$. Hence, for p in this range, a.a.s. $\delta(G) > \bar{d}$. Now we only consider $0.9 \log n/n \leq p \leq \gamma \log n/n$.

Let $\epsilon = \omega_n/\sqrt{pn}$. Note that $\beta(1 - (1/2) \log(2e)) = 1$. By Lemma 18, we have that

$$\Pr\left(\delta \leq \frac{1}{2}(1 + \epsilon)pn\right) = O\left(\exp\left(-\frac{f}{\beta} - \frac{1}{2} \log \log n + O(pn\epsilon)\right)\right) = o(1), \quad (22)$$

as $f/\beta \geq \log \log n/2 + w_n$, whereas $\epsilon pn = \omega_n \sqrt{p} = o(1)$. Moreover, $\Pr(\bar{d}/2 \geq (1 + \epsilon)pn/2) = o(1)$. Thus, a.a.s. $\delta \geq \bar{d}/2$ and so $T(G) = \delta$ by Theorem 1. On the other hand, if $f(n) \leq -\beta \log \log n/2 - \omega_n$, then $-f/\beta - \frac{1}{2} \log \log n = \omega(1 + \epsilon pn)$ and thus by Lemma 18,

$$\Pr\left(\delta > \frac{1}{2}(1 - \epsilon)pn\right) = O\left(\frac{\log n}{n} + \exp\left(+\frac{f}{\beta} + \frac{1}{2} \log \log n/2 + O(1)\right)\right) = o(1). \quad (23)$$

Moreover, $\Pr(\bar{d}/2 \leq (1 - \epsilon)pn/2) = o(1)$. Thus, a.a.s. $\delta \leq \bar{d}/2$ and thus $T(G) = \bar{d}(G)$ by Theorem 1.

6 Proof of Theorem 3

A standard tool to investigate the random graph process $G_0, \dots, G_m, \dots, G_{\binom{n}{2}}$ is the related continuous random graph process $(\mathcal{G}_p)_{p \in [0,1]}$ defined as follows. For each edge e of the complete graph with vertex set $[n]$, we associate a random variable P_e uniformly distributed in $[0, 1]$ and independent from all others. Then, for any $p \in [0, 1]$, we define \mathcal{G}_p to be the graph with vertex set $[n]$ and precisely those edges e such that $p \geq P_e$. Note that for each p , \mathcal{G}_p is distributed as $\mathcal{G}(n, p)$. This provides us with a useful way of coupling together $\mathcal{G}(n, p)$ for several values of p , since $p \leq p'$ implies $\mathcal{G}_p \subseteq \mathcal{G}_{p'}$. Moreover, let $p(m) = \min\{p \in [0, 1] : \mathcal{G}_p \text{ has at least } m \text{ edges}\}$. Then, $\mathcal{G}_{p(0)}, \dots, \mathcal{G}_{p(m)}, \dots, \mathcal{G}_{p(\binom{n}{2})}$ is distributed as $G_0, \dots, G_m, \dots, G_{\binom{n}{2}}$, since all P_e are different with probability 1. For more details on the connection between $(\mathcal{G}_p)_{p \in [0,1]}$ and $(G_m)_{0 \leq m \leq \binom{n}{2}}$ and further properties, we refer the reader to [16].

In this article, we prove several statements that hold a.a.s. simultaneously for all m in the random graph process $(G_m)_{0 \leq m \leq \binom{n}{2}}$. To do so, it is often convenient to use small bits of the continuous random graph process as follows. Given p_0 and p_1 as functions of n such that $0 \leq p_0 \leq p_1 \leq 1$, we consider $(\mathcal{G}_p)_{p_0 \leq p \leq p_1}$. Let $m_0 = m(\mathcal{G}_{p_0})$ and $m_1 = m(\mathcal{G}_{p_1})$. (Note that m_0 and m_1 are random variables with $m_0 \leq m_1$, since $\mathcal{G}_{p_0} \subseteq \mathcal{G}_{p_1}$). We colour all edges of $\mathcal{G}_{p_0} = G_{m_0}$ red and the remaining $m_1 - m_0$ edges in $\mathcal{G}_{p_1} \setminus \mathcal{G}_{p_0}$ blue. Then we can interpret $G_{m_0}, G_{m_0+1}, \dots, G_{m_1}$ as a random graph process in which we sequentially add blue edges to G_{m_0} , so that each G_m has the m_0 red edges of G_{m_0} together with the first $m - m_0$ blue edges we add in the process. This interpretation will be used many times throughout the argument.

We first prove part (ii). Instead we will prove the following stronger result.

Theorem 28. Consider the random graph process $(\mathcal{G}_p)_{0 \leq p \leq 1}$. We have that a.a.s.

(i) For all $p \leq \frac{\beta(\log n - \log \log n/2) - \omega(1)}{n-1}$, we have $\delta(\mathcal{G}_p) \leq \bar{d}(\mathcal{G}_p)/2$;

(ii) For all $p \geq \frac{\beta(\log n - \log \log n/2) + \omega(1)}{n-1}$, we have $\delta(\mathcal{G}_p) > \bar{d}(\mathcal{G}_p)/2$;

(iii) For every constant $0 < \theta < 1$ there is a constant $\rho > 0$ such that a.a.s. $\delta(\mathcal{G}_p) > \theta \bar{d}(\mathcal{G}_p)$ for all $\rho \log n / (n-1) \leq p \leq 1$.

Proof of Theorem 3 (ii). Let $p_1 = \frac{(1-\epsilon/2)\beta \log n}{n-1}$ and let $p_2 = \frac{(1+\epsilon/2)\beta \log n}{n-1}$. For $i = 1, 2$, the number of edges in $\mathcal{G}(n, p_i)$ is distributed as $\text{Bin}(\binom{n}{2}, p_i)$. By Chernoff's bound in Theorem 9, we immediately have that a.a.s. $m(\mathcal{G}(n, p_1)) \geq \binom{n}{2} p_1 + O(n) \geq (1-\epsilon)\beta n \log n/2$, and $m(\mathcal{G}(n, p_2)) \leq \binom{n}{2} p_2 + O(n) \leq (1+\epsilon)\beta n \log n/2$. Hence, a.a.s. $p(i) \leq p_1$ and $p(j) \geq p_2$ for $i = (1-\epsilon)\beta n \log n/2$ and $j = (1+\epsilon)\beta n \log n/2$. Then Theorem 3 (ii) follows from Theorem 28. \square

Proof of Theorem 28. First, we prove statement (iii). We will prove that for every $0 < \theta < 1$, there exists $\rho > 0$, such that a.a.s.

$$\delta(G_m) > \theta \bar{d}(G_m), \quad \text{for all } m \geq m_0 = (\rho/4)n \log n. \quad (24)$$

Then let $p_0 = \rho \log n / (n-1)$. By Chernoff's bound in Theorem 9, a.a.s. $m(\mathcal{G}(n, p_0)) > m_0$, i.e. a.a.s. $p(m_0) < p_0$. It follows then that a.a.s. $\delta(\mathcal{G}_p) > \theta \bar{d}(\mathcal{G}_p)$ for all $p \geq p_0$. Now we prove (24). For each m , let $\bar{p} = m / \binom{n}{2}$. Then

$$\mathbf{Pr}\left(\delta(G_m) \leq \theta \cdot 2m / (n-1)\right) = \mathbf{Pr}\left(\delta(\mathcal{G}(n, \bar{p})) \leq \theta \bar{p}n \mid m(\mathcal{G}(n, \bar{p})) = m\right).$$

By the choice of \bar{p} , $h(i) = \mathbf{Pr}(m(\mathcal{G}(n, \bar{p})) = i)$ is maximized at $i = m$. Hence, $\mathbf{Pr}(m(\mathcal{G}(n, \bar{p})) = m) \geq n^{-2}$. Thus,

$$\mathbf{Pr}\left(\delta(G_m) \leq \theta \cdot 2m / (n-1)\right) \leq \frac{\mathbf{Pr}\left(\delta(\mathcal{G}(n, \bar{p})) \leq \theta \bar{p}n\right)}{\mathbf{Pr}(m(\mathcal{G}(n, \bar{p})) = m)} \leq n^2 \mathbf{Pr}\left(\delta(\mathcal{G}(n, \bar{p})) \leq \theta \bar{p}n\right).$$

By Lemma 12 (ii), for every $0 < \theta < 1$, we can choose $\rho > 0$ sufficiently large such that the above is less than $1/n^6$ for every $m \geq (\rho/4)n \log n$ (correspondingly $\bar{p} \geq (\rho/2) \log n / (n-1)$). Hence, with probability at least $1 - n^{-1}$, claim (24) is true.

Next we prove statements (i) and (ii). Let $f = o(\log n)$ be any function that goes to ∞ as $n \rightarrow \infty$. Let $p_i = (\beta(\log n - \log \log n/2) - f^2 - if) / (n-1)$ and let $q_i = (\beta(\log n - \log \log n/2) + f^2 + if) / (n-1)$, for all $i \geq 1$. Let T be the maximum integer such that $p_T \geq \log n / 2(n-1)$ and redefine $p_T = \log n / 2(n-1)$. Let ρ be the constant satisfying statement (iii) with $\theta = 3/4$. Let T' be the maximum integer such that $q_{T'} \leq \rho \log n / (n-1)$ and redefine $q_{T'} = \rho \log n / (n-1)$. Obviously, $T, T' = O(\log n)$.

Claim 2. There exists a positive constant C , such that, for every $1 \leq i < T$,

$$\mathbf{Pr}\left(\delta(\mathcal{G}_{p_i}) > \bar{d}(\mathcal{G}_{p_{i+1}})/2\right) \leq C(f^{-i} + \log n/n),$$

and for every $1 \leq i < T'$,

$$\mathbf{Pr}\left(\delta(\mathcal{G}_{q_i}) \leq \bar{d}(\mathcal{G}_{q_{i+1}})/2\right) \leq C(f^{-i} + n^{-1}).$$

By Lemma 12 (i), a.a.s. for all $p < p_T = \log n/2(n-1)$, $\delta(\mathcal{G}_p) = 0$ and thus $\delta(\mathcal{G}_p) \leq \bar{d}(\mathcal{G}_p)/2$ holds. By Claim 2, with probability at least

$$1 - \sum_{1 \leq i < T} C(f^{-i} + \log n/n) = 1 - o(1),$$

for all $1 \leq i < T$ and for all $p_{i+1} \leq p \leq p_i$,

$$\delta(\mathcal{G}_p) \leq \delta(\mathcal{G}_{p_i}) \leq \bar{d}(\mathcal{G}_{p_{i+1}})/2 \leq \bar{d}(\mathcal{G}_p)/2.$$

Thus, a.a.s. $\delta(\mathcal{G}_p) \leq \bar{d}(\mathcal{G}_p)/2$ for all $p \leq \frac{\beta(\log n - \log \log n/2) + \omega(1)}{n-1}$, since f is an arbitrary function in $\omega(1)$.

By (iii) (with $\theta = 3/4$), we only need to prove that a.a.s. $\delta(\mathcal{G}_p) > \bar{d}(\mathcal{G}_p)/2$ for all $p \geq \frac{\beta(\log n - \log \log n/2) + \omega(1)}{n-1}$ and $p \leq q_{T'} = \rho \log n/(n-1)$. Similar to the previous argument, with probability at least

$$1 - \sum_{i \geq 1} C(f^{-i} + n^{-1}) = 1 - o(1),$$

for all $1 \leq i < T'$ and for every p with $q_i \leq p \leq q_{i+1}$,

$$\delta(\mathcal{G}_p) \geq \delta(\mathcal{G}_{q_i}) > \bar{d}(\mathcal{G}_{q_{i+1}})/2 \geq \bar{d}(\mathcal{G}_p)/2.$$

Thus, a.a.s. $\delta(\mathcal{G}_p) > \bar{d}(\mathcal{G}_p)/2$ for all $p \geq \frac{\beta(\log n - \log \log n/2) + \omega(1)}{n-1}$. \square

Proof of Claim 2. In this proof the asymptotic statements are uniform for all p_i . By Lemma 10, for $\sigma = n^{-1/3}$ and $A = 1/12$,

$$\Pr(|\bar{d}(\mathcal{G}_{p_i}) - p_i n| > \sigma p_i n) = \exp(-A\sigma^2 n^2 p_i) \leq n^{-1}.$$

Hence, by Lemma 18 with $\epsilon = -\Theta(f/\log n)$ and the function f in the Lemma 18 as $-\beta \log \log n/2 - f^2 - if$ and noting that $\epsilon p n = O(f)$, for every $1 \leq i < T$,

$$\begin{aligned} \Pr(\delta(\mathcal{G}_{p_i}) > \bar{d}(\mathcal{G}_{p_{i+1}})) &\leq \Pr(\delta(\mathcal{G}_{p_i}) > p_{i+1}n/2 - \sigma p_{i+1}n) + \Pr(\bar{d}(\mathcal{G}_{p_{i+1}}) < p_{i+1}n/2 - \sigma p_{i+1}n) \\ &= O\left(\frac{\log n}{n} + \exp\left(\frac{-f^2 - if}{\beta} + O(f)\right)\right) = O(f^{-i} + \log n/n). \end{aligned}$$

Similarly, for every $1 \leq i < T'$, by Lemma 18 with $\epsilon = -\Theta(f/\log n)$ and $f(n) = -\beta \log \log n/2 + f^2 + if$ and noting that $\epsilon p n = O(f)$,

$$\begin{aligned} \Pr(\delta(\mathcal{G}_{q_i}) \leq \bar{d}(\mathcal{G}_{q_{i+1}})) &\leq \Pr(\delta(\mathcal{G}_{q_i}) \leq q_{i+1}n/2 + \sigma q_{i+1}n) + \Pr(\bar{d}(\mathcal{G}_{q_{i+1}}) > q_{i+1}n/2 + \sigma q_{i+1}n) \\ &= O\left(\exp\left(\frac{-f^2 - if}{\beta} + O(f)\right)\right) = O(f^{-i} + 1/n). \end{aligned}$$

\square

We now proceed to prove statement (i) of Theorem 3. For any graph G , define $t(G) = \min\{\delta(G), \bar{d}(G)/2\}$. First, define $p_0 = 0.9 \log n/(n-1)$, $p_1 = \gamma_1 \log n/(n-1)$ and $p_2 = \gamma_2 \log n/(n-1)$, for some constants $1 < \gamma_1 < \gamma_2$ that we specify later. We prove the statement separately for $(\mathcal{G}_p)_{p_0 \leq p \leq p_1}$, $(\mathcal{G}_p)_{p_1 \leq p \leq p_2}$ and $(\mathcal{G}_p)_{p_2 \leq p \leq 1}$. For $(\mathcal{G}_p)_{0 \leq p \leq p_0}$ it is trivially true since a.a.s. $\delta(\mathcal{G}_p) = 0$ for all $0 \leq p \leq p_0$ by Lemma 12 (i).

Part 1 ($p_0 \leq p \leq p_1$): Let $\epsilon > 0$ be a constant chosen to satisfy Lemma 22 for $\gamma = 1.1$. Pick a sufficiently small constant $1 < \gamma_1 < 1.1$ and recall $p_0 = 0.9 \log n / (n - 1)$ and $p_1 = \gamma_1 \log n / (n - 1)$. From Lemma 10, a.a.s.

$$\bar{d}(\mathcal{G}_{p_1}) \leq (4/3)\bar{d}(\mathcal{G}_{p_0}). \quad (25)$$

Moreover, in view of Lemma 12 (ii), we assume that γ_1 is small enough so that a.a.s.

$$\delta(\mathcal{G}_{p_1}) \leq (\epsilon/16)\bar{d}(\mathcal{G}_{p_1}) \leq (\epsilon/12)\bar{d}(\mathcal{G}_{p_0}). \quad (26)$$

Colour edges in \mathcal{G}_{p_1} so that all edges in \mathcal{G}_{p_0} are coloured red and all edges in $\mathcal{G}_{p_1} \setminus \mathcal{G}_{p_0}$ are coloured blue, as described in the beginning of the section. For each vertex $v \in \mathcal{G}_{p_1}$, the red (blue) degree of v is the number of red (blue) edges incident with v . Let S be the set of ϵ -light vertices of \mathcal{G}_{p_0} . Since $\delta(\mathcal{G}_{p_0}) = 0$ a.a.s., the ϵ -light vertices are a.a.s. precisely those vertices with degree at most $\epsilon\bar{d}(\mathcal{G}_{p_0})$ in \mathcal{G}_{p_0} . By the choice of ϵ and Lemma 22, a.a.s. the vertices in S have no internal edges and no common neighbours in \mathcal{G}_{p_0} . By Lemma 23, a.a.s. this remains true for S in the whole process $(\mathcal{G}_p)_{p_0 \leq p \leq p_1}$ as blue edges are added.

For each $p_0 \leq p \leq p_1$, let S_p be the set of $(11\epsilon/16)$ -light vertices of \mathcal{G}_p (i.e. vertices of degree at most $\delta(\mathcal{G}_p) + (11/16)\epsilon\bar{d}(\mathcal{G}_p)$ in \mathcal{G}_p). Note that a.a.s. S contains S_p for all p in this range, since for any $v \in S_p$, its degree in \mathcal{G}_{p_0} (i.e. the red degree of v in \mathcal{G}_p) is at most

$$\delta(\mathcal{G}_p) + (11/16)\epsilon\bar{d}(\mathcal{G}_p) \leq \delta(\mathcal{G}_{p_1}) + (11/16)\epsilon\bar{d}(\mathcal{G}_{p_1}) \leq (\epsilon/12)\bar{d}(\mathcal{G}_{p_0}) + (11/12)\epsilon\bar{d}(\mathcal{G}_{p_0}) = \epsilon\bar{d}(\mathcal{G}_{p_0}),$$

where we used (25) and (26).

We just showed that a.a.s. in $(\mathcal{G}_p)_{p_0 \leq p \leq p_1}$ the set of $(11\epsilon/16)$ -light vertices of \mathcal{G}_p have no internal edges and no common neighbours. Moreover, from (26) and by monotonicity of $\delta(\mathcal{G}_p)$ and $\bar{d}(\mathcal{G}_p)$ with respect to p , we have that a.a.s.

$$\delta(\mathcal{G}_p) \leq \delta(\mathcal{G}_{p_1}) \leq (\epsilon/12)\bar{d}(\mathcal{G}_{p_0}) \leq (\epsilon/12)\bar{d}(\mathcal{G}_p)$$

in the whole process $(\mathcal{G}_p)_{p_0 \leq p \leq p_1}$. Putting all that together, we have that a.a.s. conditions of Lemma 26 are satisfied in $(\mathcal{G}_p)_{p_0 \leq p \leq p_1}$ (replacing ϵ by $(11/16)\epsilon$), and therefore a.a.s. $T(\mathcal{G}_p) = \delta(\mathcal{G}_p)$ simultaneously for all p in this range.

Part 2 ($p_1 \leq p \leq p_2$): Recall that $p_1 = \gamma_1 \log n / (n - 1)$ and $p_2 = \gamma_2 \log n / (n - 1)$, where γ_1 is as in Part 1, and $\gamma_2 > \gamma_1$ is a sufficiently large constant. In view of Theorem 28 (iii), we assume that γ_2 is large enough so that a.a.s. $\delta(\mathcal{G}_p) \geq (3/4)\bar{d}(\mathcal{G}_p)$ in the whole process $(\mathcal{G}_p)_{p_2 \leq p \leq 1}$.

Define $q_i = (1 + 1/\log n)^i p_1$ for each $i = 0, 1, 2, \dots$, and let T be the smallest integer such that $q_T \geq p_2$. Redefine $q_T = p_2$. We have $T \leq 2 \log(\gamma_2/\gamma_1) \log n = O(\log n)$, since eventually

$$(1 + 1/\log n)^{2 \log(\gamma_2/\gamma_1) \log n} > \gamma_2/\gamma_1.$$

To prove the statement for $(\mathcal{G}_p)_{p_1 \leq p \leq p_2}$, it suffices to see that for every $0 \leq i \leq T - 1$, we have $T(\mathcal{G}_p) = t(\mathcal{G}_p)$ throughout the process $(\mathcal{G}_p)_{q_i \leq p \leq q_{i+1}}$ with probability at least $1 - 1/\log^2 n$, and then simply take a union bound over all i .

Let ϵ be as in Lemma 22 (putting $\gamma = 2\gamma_2$), and fix $0 \leq i \leq T - 1$. We verify that with probability at least $1 - 1/\log^2 n$ all conditions (a')–(e') of Lemma 27 are satisfied in

$(\mathcal{G}_p)_{q_i \leq p \leq q_{i+1}}$. We colour as before the edges of \mathcal{G}_{q_i} red, and the additional edges in $\mathcal{G}_{q_{i+1}} \setminus \mathcal{G}_{q_i}$ blue.

Let S be the set of vertices that are ϵ -light in \mathcal{G}_{q_i} (they have red degree at most $\delta(\mathcal{G}_{q_i}) + \epsilon \bar{d}(\mathcal{G}_{q_i})$). For each $q_i \leq p \leq q_{i+1}$, define S_p to be the set of vertices that are $\epsilon/2$ -light in \mathcal{G}_p . From Lemma 10 and Lemma 20 we have that with probability at least $1 - e^{-C \log n}$,

$$\bar{d}(\mathcal{G}_{q_i}) \sim \bar{d}(\mathcal{G}_{q_{i+1}}) \quad \text{and} \quad \delta(\mathcal{G}_{q_i}) \sim \delta(\mathcal{G}_{q_{i+1}}). \quad (27)$$

These equations imply that $S \supseteq S_p$ for all p in our range, since the red degree of any vertex in S_p is at most

$$\delta(\mathcal{G}_p) + (\epsilon/2) \bar{d}(\mathcal{G}_p) \leq \delta(\mathcal{G}_{q_{i+1}}) + (\epsilon/2) \bar{d}(\mathcal{G}_{q_{i+1}}) \sim \delta(\mathcal{G}_{q_i}) + (\epsilon/2) \bar{d}(\mathcal{G}_{q_i}) \leq \delta(\mathcal{G}_{q_i}) + \epsilon \bar{d}(\mathcal{G}_{q_i}).$$

By Lemma 23, with probability at least $1 - e^{-C \log n}$ the vertices in S do not get common neighbours or internal edges as the blue edges are added in $(\mathcal{G}_p)_{q_i \leq p \leq q_{i+1}}$. This implies condition (a') replacing ϵ by $\epsilon/2$. Condition (b') holds with probability at least $1 - e^{-C(\log n)^{1/3}}$ by Lemma 11 for \mathcal{G}_{q_i} and thus also for all $p \in [q_i, q_{i+1}]$ due to (27). The same holds for Condition (c') with Lemma 13 with probability at least $1 - e^{-C \log n}$. By Lemma 19, there exists $\sigma > 0$ such that uniformly, for all $p \in [p_1, p_2]$, we have that $\delta(\mathcal{G}_p) \geq \sigma p n$ with probability at least $1 - e^{-C \log n}$, which implies that $t(\mathcal{G}_p) \geq \sigma' p n$ for a positive constant σ' for all $p \in [p_1, p_2]$. Then condition (e') holds with probability at least $1 - e^{-C \log n}$ for all $\mathcal{G}_{q_{i+1}}$ by Lemma 14 and so for all $p \in [q_i, q_{i+1}]$ due to (27). Condition (d') holds with by Lemma 24 with probability $1 - e^{-C \log n}$.

Taking the union bound for all $0 \leq i \leq T-1$, we have that a.a.s. $T(\mathcal{G}_p) = t(\mathcal{G}_p)$ throughout the process $(\mathcal{G}_p)_{p_1 \leq p \leq p_2}$ by Lemma 27.

Part 3 ($p_2 \leq p \leq 1$): Let γ_2 be as in Part 2, and $p_2 = \gamma_2 \log n / (n-1)$. Recall from the definition of γ_2 that a.a.s. $\delta(\mathcal{G}_p) \geq (3/4) \bar{d}(\mathcal{G}_p)$ in the whole process $(\mathcal{G}_p)_{p_2 \leq p \leq 1}$, and therefore Condition (a') in Lemma 27 holds. Define $q_i = (1 + 1/\log n)^i p_2$ for each $i = 0, 1, 2, \dots$, and let T be the smallest integer such that $q_T \geq 1$. Redefine $q_T = 1$. Observe that $T \leq 3 \log^2 n$, since eventually $(1 + 1/\log n)^{3 \log^2 n} \geq n^{2.5}$. The same argument as in Part 2 shows that for every $0 \leq i \leq T-1$, Conditions (b')-(e') in Lemma 27 are satisfied throughout the process $(\mathcal{G}_p)_{q_i \leq p \leq q_{i+1}}$ with probability at least $1 - 1/\log^3 n$. Taking the union bound over all i , we conclude that a.a.s. all condition in Lemma 27 hold and therefore $T(\mathcal{G}_p) = \bar{d}(\mathcal{G}_p)/2$, during the whole process $(\mathcal{G}_p)_{p_2 \leq p \leq 1}$.

7 Proof of Theorem 5

We first prove statement (b). In view of Theorem 3, we assume that $T(G_m) = \min\{\delta(G_m), \lfloor m/(n-1) \rfloor\}$ for all $m = 0, 1, \dots, \binom{n}{2}$. Then we pick any m such that $\delta(G_m) \geq \bar{d}(G_m)/2 = m/(n-1)$. If $n-1$ divides m , then $T(G_m) = m/(n-1)$ and thus $A(G_m) = m/(n-1)$. If $n-1$ does not divide m , let m' be the smallest integer $m' > m$ divisible by $n-1$. Since, the minimum degree is always an integer, we have $\delta(G_m) \geq \lceil m/(n-1) \rceil = m'/(n-1)$. Moreover, G_m is a spanning subgraph of $G_{m'}$ and thus $\delta(G_{m'}) \geq \delta(G_m) \geq m'/(n-1)$. Therefore, from our assumption on the random graph process, we have $T(G_{m'}) = m'/(n-1) = \lceil m/(n-1) \rceil$, and

these $\lceil m/(n-1) \rceil$ edge-disjoint spanning trees cover all edges of G_m , so $A(G_m) = \lceil m/(n-1) \rceil$. This completes the proof of the statement.

Next we proceed to prove statement (a). Let f be any function of n such that $f \rightarrow \infty$ arbitrarily slowly and $f = o(\log n)$. Define $p_j = (1 + 1/f)^j f/n$ for each $j = 0, 1, 2, \dots$, and let T be the largest integer such that $p_T \leq \frac{\beta \log n}{(1-\epsilon/2)n}$.

Claim 3. for every $0 \leq j < T$ and for every m such that $p_j \leq p(m) \leq p_{j+1}$, the bound $A(\mathcal{G}_{p(m)}) = A(G_m) \leq \lceil \frac{m+\phi_2}{n-1} \rceil$ holds in the random graph process $(\mathcal{G}_p)_{p \in [p_j, p_{j+1}]}$ with probability at least $1 - 1/(p_j n)^2$.

Assuming Claim 3, the probability that $A(G_m) \leq \lceil \frac{m+\phi_2}{n-1} \rceil$ fails somewhere in the random graph process between $\mathcal{G}(n, p_0)$ and $\mathcal{G}(n, p_T)$ is at most

$$\frac{1}{f^2} \sum_{j=0}^{T-1} (1 + 1/f)^{-2j} \leq \frac{1}{f^2} \sum_{j=0}^{\infty} (1 + 1/f)^{-j} = \frac{1 + 1/f}{f} = o(1). \quad (28)$$

Moreover, we have that $p_T \geq \frac{\beta \log n}{(1-\epsilon/4)n}$ eventually (for $n > n_0$ depending only on f). Then there is $\sigma > 0$ such that a.a.s. $m > (1 + \sigma)\beta n \log n$ for every m with $p(m) \geq p_T$. Then, by Theorem 3 (ii), a.a.s. $\delta(G_m) > \bar{d}(G_m)/2$ for all m with $p(m) \geq p_T$. Thus, we only need to restrict our discussion to p_j with $j \leq T$.

Similarly, let T' be the largest integer such that $p_{T'} \leq \frac{\beta \log n}{(1+\epsilon/2)n}$.

Claim 4. for every $0 \leq j < T'$ and for every m such that $p(m) \in [p_j, p_{j+1}]$, the bound $A(G_m) \geq \lceil \frac{m+\phi_1}{n-1} \rceil$ holds in the random graph process $(\mathcal{G}_p)_{p \in [p_j, p_{j+1}]}$ with probability at least $1 - 1/(p_j n)^2$.

By the same argument as in (28), assuming Claim 2, the probability that $A(G_m) \geq \lceil \frac{m+\phi_1}{n-1} \rceil$ fails somewhere in the random graph process $(\mathcal{G}_p)_{p \in [p_0, p_{T'}]}$ is also $o(1)$. Moreover, note that since $p_{T'} \geq \frac{\beta \log n}{(1+2\epsilon/3)n}$, then a.a.s. for every m with $p(m) \in [p_{T'}, 1]$, we have $2m/n \geq \frac{\beta \log n}{(1+3\epsilon/4)}$ by Lemma 10, and therefore eventually $\phi_1 \leq 1/2$ for all m in this range. Hence, for all m not divisible by $n-1$, we eventually have

$$A(G_m) \geq \left\lceil \frac{m}{n-1} \right\rceil = \left\lceil \frac{m+1/2}{n-1} \right\rceil = \left\lceil \frac{m+\phi_1}{n-1} \right\rceil.$$

Otherwise, for m divisible by $n-1$, the condition $\delta(G_m) < \bar{d}(G_m)/2 = m/(n-1)$ implies $A(G_m) > m/(n-1)$, since we cannot have a full factorisation of G_m into $m/(n-1)$ spanning trees, so then

$$A(G_m) \geq \frac{m}{n-1} + 1 = \left\lceil \frac{m+1/2}{n-1} \right\rceil = \left\lceil \frac{m+\phi_1}{n-1} \right\rceil.$$

Putting everything together, we showed that a.a.s. (1) holds simultaneously for all G_m in the random graph process \mathcal{G}_p for p between f/n and 1. Given any $m_0 = \omega(n)$ as in the statement, we may simply choose $f = m_0/n$. Then a.a.s. $m(\mathcal{G}_{f/n}) = m(\mathcal{G}(n, f/n)) \leq (3/4)fn = (3/4)m_0$, and statement (a) holds for the desired range of m . It only remains to prove Claims 3 and 4.

Proof of Claim 3. In this proof the asymptotic statements are uniform for all p_j and depend only on f . Given any $0 \leq j < T$, we consider the random graph process $(\mathcal{G}_p)_{p \in [p_j, p_{j+1}]}$. Define $\delta_j = \delta(\mathcal{G}_{p_j})$, $\bar{d}_j = \bar{d}(\mathcal{G}_{p_j})$, $m_j = m(\mathcal{G}_{p_j})$, $t_j = \min\{\delta_j, \bar{d}_j/2\}$. By Lemma 10, we have $\bar{d}_j \sim \bar{d}_{j+1} \sim np_j$ with probability at least $1 - Ce^{-p_j n}$ for a positive constant C .

Let $\hat{\epsilon} > 0$ be a sufficiently small constant so that

$$e^{-1} \left(\frac{2e}{1+2\hat{\epsilon}} \right)^{(1+2\hat{\epsilon})/2} < e^{-(1-\epsilon)/\beta} \quad \text{and} \quad e^{-1} \left(\frac{2e}{1-\hat{\epsilon}} \right)^{(1-\hat{\epsilon})/2} > e^{-(1+\epsilon/4)/\beta}. \quad (29)$$

Colour the edges of $G_{m_j} = \mathcal{G}_{p_j}$ red. Let $G_{m_{j+1}} = \mathcal{G}_{p_{j+1}}$. Colour edges in $G_{m_{j+1}} \setminus G_{m_j}$ blue. For any vertex $v \in G_{m_{j+1}}$, define the red (blue) degree of v to be the number of red (blue) edges that are incident with v . Call a vertex light if its red degree is at most $\frac{(1+2\hat{\epsilon})}{2} \bar{d}_{j+1}$. A vertex is called heavy if its red degree is at least $(3/4) \bar{d}_{j+1}$. The vertices that are neither light nor heavy are called medium vertices. We have that for any constant $\alpha > 0$, with probability $1 - e^{-p_j n}$, by Lemma 10 and Lemma 25,

$$\left| m_{j+1} - p_{j+1} \binom{n}{2} \right| < \alpha p_{j+1} n^2, \quad \Delta(G_{m_{j+1}}) = O(\log n), \quad (30)$$

where $\Delta(G)$ denotes the maximum degree of G . By Lemma 15(i) with $k = \frac{1+2\hat{\epsilon}}{2} \bar{d}_{j+1}$, the expected number of light vertices is at most

$$n \left(\frac{1}{e} \left(\frac{2e}{1+2\hat{\epsilon}} \right)^{\frac{1+2\hat{\epsilon}}{2}} + o(1) \right)^{pn}.$$

Thus by Markov's inequality and (29), the number of light vertices is

$$\ell \leq \frac{n}{\bar{d}_{j+1} \exp \left(\frac{(1-\epsilon) 2m_{j+1}}{\beta n} \right)} = o(n), \quad (31)$$

with probability at least $1 - e^{-D p_j n}$, for a positive constant D . Similarly, by Lemma 15(i) with $k = \frac{3}{4} \bar{d}_{j+1}$, with probability $1 - e^{-D p_j n}$ for a positive constant D , the number of heavy vertices is

$$h = n - o(n). \quad (32)$$

For the following construction, assume (30), (31) and (32) hold. We add $g = \bar{d}_{j+1} \ell$ new edges (different from the previous red and blue edges) to $G_{m_{j+1}}$, which we colour green, in such a way that every light vertex is incident with exactly \bar{d}_{j+1} green edges; every heavy vertex is incident to at most one green edge; and no green edge is incident to any medium vertices. (So green edges only connect light and heavy vertices.) This can be done by (31), since the total number of green edges

$$g \leq \frac{n}{\exp \left(\frac{(1-\epsilon) 2m_{j+1}}{\beta n} \right)}$$

is much smaller than the number of heavy vertices eventually. Finally, greedily add n yellow edges to $G_{m_{j+1}}$, different to all previous red, blue and green ones, in a way that each yellow edge connects two heavy vertices and each heavy vertex is incident with at most 3 yellow edges (this can be done greedily since we have $h \sim n$ heavy vertices by (32) and the maximum degree (adding red, blue and green degrees together) is $O(\log n)$ by (30)).

We may regard the sequence of graphs $G_{m_j} \subset G_{m_{j+1}}, \dots \subset G_{m_{j+1}}$ as a process in which we sequentially add blue edges to G_{m_j} , so the edges of each G_m are precisely the red ones

together with the first $m - m_j$ blue ones. For each m in our range, we define G'_m as $E(G'_m) = G_m \cup E_g \cup E_y$, where E_g is the set of green edges added to $G_{m_{j+1}}$ and E_y is an arbitrary subset of yellow edges added to $G_{m_{j+1}}$ so that the number of edges of the resulting graph G'_m is a multiple of $n - 1$. We now verify that $G'_{m_j}, \dots, G'_{m_{j+1}}$ satisfy all conditions (a')–(e') of Lemma 27, assuming that (30), (31) and (32) and some additional events hold. We give bounds on the probabilities of these events.

First observe that for all $m_j \leq m \leq m_{j+1}$, we have $\bar{d}(G'_m) \leq \frac{2(m_{j+1}+g+n)}{n-1} \leq \bar{d}_{j+1} + 2$ and similarly $\bar{d}(G'_m) \geq \frac{2m_j}{n-1} = \bar{d}_j$, so

$$\bar{d}(G'_m) \sim \bar{d}_{j+1} \sim \bar{d}_j \quad (33)$$

Hence, for all m in the range,

$$\delta(G'_m) \geq \frac{(1 + 2\hat{\epsilon})\bar{d}_{j+1}}{2} \geq \frac{(1 + \hat{\epsilon})\bar{d}(G'_m)}{2}, \quad (34)$$

so (a') holds and $\delta = \Omega(\bar{d}) = \omega(1)$. For any S , let $d_r(S)$ denote the average red degree of S . By Lemma 11, with probability $1 - e^{-C(p_j n)^{1/3}}$ for a positive constant C , for all S with $S \geq \zeta n$, $d_r(S) \geq \bar{d}(G_{m_j})(1 - o(1)) \geq \bar{d}(G'_m)(1 - o(1))$ by (33). Thus, (b') holds by noting that $d(S) \geq d_r(S)$. By applying Lemma 13 to the red edges and using (33), we deduce condition (c') holds with probability at least $1 - e^{-Cp_j n^2}$ for all G'_m .

For (e'), first note that $t(G'_{m_j}) = \Omega(\bar{d}_{j+1})$. Then, Lemma 14 applied to $G_{m_{j+1}}$ (i.e. only red and blue edges) shows that with probability $1 - Ce^{-(p_j n)^2}$ all sets S of size $s < \zeta n$ have at most $(\hat{\epsilon}/4)t(G'_{m_j})s$ red and blue edges inside. Let G'' be obtained by adding all g green edges and all n yellow edges. So $G'_m \subseteq G''$ for all $m_j \leq m \leq m_{j+1}$. We bound the number of internal edges inside S in G'' . Let $s_2 \leq s$ be the number of heavy vertices in S . Each internal green edge in S must be incident to one of the s_2 heavy vertices inside, and the number of yellow edges in S is at most $3s_2$, since each heavy vertex is incident to at most 3 of them. Therefore, the number of green and yellow edges in S is at most $4s_2 \leq 4s$, and the total number of edges inside S in G'' (and thus, in all G'_m , $m_j \leq m \leq m_{j+1}$) is at most $(\hat{\epsilon}/4)t(G'_{m_j})s + 4s \leq (\hat{\epsilon}/2)t(G'_{m_j})s$, since $t(G'_{m_j}) \rightarrow \infty$. Thus (e') holds for all $m_j \leq m \leq m_{j+1}$, since $t(G'_{m_j}) \leq t(G'_m)$.

Finally, we prove (d'): Let S be a set with $1 \leq |S| \leq n/2$ (otherwise we take \bar{S}). Suppose first that $1 \leq |S| \leq \zeta n$. From what we proved before for (e') and (34), the number of internal edges in S is at most $(\hat{\epsilon}/2)t(G'_{m_j})s$ and so, for each G'_m ,

$$E(S, \bar{S}) \geq \delta(G'_m)s - (\hat{\epsilon}/2)t(G'_{m_j})s \geq \frac{(1 + \hat{\epsilon})\bar{d}(G'_{m_j})}{2}s - (\hat{\epsilon}/2)t(G'_{m_j})s \geq t(G'_{m_j})s \geq t(G'_m).$$

Otherwise, if $\zeta n \leq |S| \leq n/2$, (c') gives us what we need using only red edges.

Hence, in view of Lemma 27, with probability $1 - e^{-C(p_j n)^{1/3}} \geq 1 - 1/(p_j n)^2$ since $p_j n \geq f = \omega(1)$, for all $m_j \leq m \leq m_{j+1}$, we have

$$T(G'_m) = m(G'_m)/(n - 1) = \lceil (m + g)/(n - 1) \rceil \leq \lceil (m + \phi_2)/(n - 1) \rceil,$$

since by construction $m(G'_m)$ is the smallest integer that is at least $m + g$ and is divisible by $n - 1$.

This implies the claim since

$$A(G_m) \leq A(G'_m) = m(G'_m)/(n-1) \leq \lceil (m + \phi_2)/(n-1) \rceil.$$

□

Proof of Claim 4. We pick a constant $\hat{\epsilon} > 0$ as in (29). By (30) and the definition of p_j , with probability $1 - e^{-p_j n}$, for all $m_j \leq m \leq m_{j+1}$ we have

$$A(G_m) \geq \lceil m/(n-1) \rceil \geq \bar{d}(G_{m_j})/2 > (1 - \hat{\epsilon})\bar{d}(G_{m_{j+1}})/2 \quad (35)$$

and the maximum degree of G_m is $O(\log n)$. Let us redefine light vertices of $G_{m_{j+1}}$ to be vertices with degree (red degree plus blue degree) at most $(1 - \hat{\epsilon})\bar{d}(G_{m_{j+1}})/2$. By Lemma 15(i) with $k = \frac{(1-\hat{\epsilon})}{2}\bar{d}(G_{m_{j+1}})$ and (29), the expected number of light vertices in $G_{m_{j+1}}$ is at least

$$n \sqrt{\frac{1}{k}} \left(\frac{1}{e} \left(\frac{2e}{(1-\hat{\epsilon})} \right)^{\frac{1-\hat{\epsilon}}{2}} + o(1) \right)^{pn} \geq n e^{-(1+\epsilon'/4)pn/\beta} \geq 2n e^{-\frac{(1+\epsilon/4)}{\beta} \frac{2m_j}{n}},$$

for some constant $\epsilon' < \epsilon$ and the inequality holds since $f = o(\log n)$. Note that with probability at least $1 - e^{-C \log n}$, for all $j \leq T'$, $2m_j \leq (1 + \epsilon/8)\beta n \log n$ and so

$$\frac{(1 + \epsilon/4)}{\beta} \frac{2m_j}{n} \leq \frac{(1 + \epsilon/4)(1 + \epsilon/8)}{1 + \epsilon/2} \log n \leq \sigma \log n,$$

for some $0 < \sigma < 1$, depending only on ϵ . Then, by Lemma 16 and by Chebyshev's inequality, there are

$$\ell' \geq \frac{n}{\exp \left(\frac{(1+\epsilon/4)}{\beta} \frac{2m_j}{n} \right)}$$

light vertices in $G_{m_{j+1}}$ with probability at least $1 - O(\log n/n) - n^{\sigma-1} \geq 1 - 1/(p_j n)^2$. By (35), with probability at least $1 - e^{-p_j n}$, these light vertices have degree in G_m strictly less than $A(G_m)$ for all m in the range $m_j \leq m \leq m_{j+1}$ since $G_m \subseteq G_{m_{j+1}}$.

For each G_m we construct G'_m as follows. Let \mathcal{F}_m be the set of $A(G_m)$ edge-disjoint forests covering G_m . For every light vertex v and for every forest $F \in \mathcal{F}_m$, if F has no edge incident to v , then we add a new edge connecting v to some non-light vertex, and make this new edge be part of F (this can always be done since both $|\mathcal{F}_m|$ and the maximum degree are $O(\log n)$ and the number of non-light vertices is $n - o(n)$). Observe that G'_m has at least $m + \ell'$ edges since for each light vertex v we added at least one edge as the degree of v is less than $|\mathcal{F}_m| = A(G_m)$. By construction, G'_m and G_m have the same arboricity, so

$$A(G_m) = A(G'_m) \geq \lceil (m + \ell')/(n-1) \rceil \geq \lceil (m + \phi_1)/(n-1) \rceil.$$

□

8 Proof of Theorem 4

Lemma 29. *Let $G \sim \mathcal{G}(n, p)$, where $p \leq c/n$ for some constant $c > 0$. Then there exists another constant $\alpha > 0$, such that a.a.s. all subgraphs of G with order at most αn have average degree at most 2.2.*

Proof. Let X_s denote the number of sets $S \subseteq [n]$ with $|S| = s$ and $|E[S]| > 1.1s$. Let $r = s/n$. Then

$$\mathbf{E}X_s \leq \binom{n}{s} \binom{s^2}{1.1s} \left(\frac{c}{n}\right)^{1.1s} \leq \left(\frac{e}{r} \left(\frac{ern}{1.1} \frac{c}{n}\right)^{1.1}\right)^s = (Cr^{0.1})^s,$$

where $C = e^{2.1}c^{1.1}/1.1^{1.1}$ is a constant depending only on c . Thus, by choosing α sufficiently small, we have that for all $r \leq \alpha$, $Cr^{0.1} < 1/2$. It follows then that

$$\sum_{1 \leq s \leq \alpha n} \mathbf{E}X_s = \sum_{1 \leq s \leq \log n} \mathbf{E}X_s + \sum_{\log n < s \leq \alpha n} \mathbf{E}X_s = O\left(\frac{\log^{0.1} n}{n^{0.1}} + 2^{-\log n}\right) = o(1).$$

□

For any integer $k \geq 0$ and real $\mu \geq 0$, define

$$f_k(\mu) = e^{-\mu} \sum_{i \geq k} \frac{\mu^i}{i!}.$$

For any $k \geq 3$, define $h_k(\mu) = \frac{\mu}{f_{k-1}(\mu)}$, and let

$$c_k = \inf\{h_k(\mu), \mu > 0\}, \quad \forall k \geq 3 \quad \text{and} \quad c_2 = 1. \quad (36)$$

For any $c > c_k$, define $\mu_{c,k}$ to be the larger solution of $h_k(\mu) = c$.

The following theorem follows from a result about the threshold for the appearance of a giant k -core, first proved by Pittel, Spencer and Wormald [24], and later reproved by many authors. See [17, 15, 19].

Theorem 30. *Let $k \geq 2$ be fixed and let c_k be defined as in (36). Then for all $c > c_k$, a.a.s. $\mathcal{G}(n, c/n)$ has a non-empty k -core with $f_k(\mu_{c,k})n + o(n)$ vertices and $\frac{1}{2}\mu_{c,k}f_{k-1}(\mu_{c,k})n + o(n)$ edges. For all $k \geq 3$ and $c < c_k$, a.a.s. $\mathcal{G}(n, c/n)$ has an empty k -core.*

Cain, Sanders and Wormald [4] proved that for every $k \geq 2$ and $\epsilon > 0$, a.a.s. we have that if the average degree of the $(k+1)$ -core of $\mathcal{G}(n, p)$ is at most $2k - \epsilon$, then $\mathcal{G}(n, p)$ is k -orientable: all its edges can be oriented so that no vertex has indegree more than k . On the other hand, Hakimi's characterisation [13] tells that a graph is k -orientable if and only if it contains no subgraph whose average degree is more than $2k$. These two results immediately imply the following theorem.

Theorem 31. *Given any positive integer $k \geq 2$ and an arbitrarily small $\epsilon > 0$, a.a.s. we have that if the average degree of the $(k+1)$ -core of $\mathcal{G}(n, p)$ is at most $2k - \epsilon$, then there is no subgraph of $\mathcal{G}(n, p)$ whose average degree is more than $2k$.*

Corollary 32. *Given any positive integer $k \geq 2$, a.a.s. we have that if the average degree of the $(k+1)$ -core of $\mathcal{G}(n, p)$ is at most $2k + o(1)$, then there is no subgraph of $\mathcal{G}(n, p)$ whose average degree is more than $2k + o(1)$.*

Proof. It is easy to verify that $\mu f_{k-1}(\mu)/f_k(\mu)$ is a strictly increasing function of μ , which goes to infinity as $\mu \rightarrow \infty$. It is also easy to verify that $h_k(\mu)$ is a strictly increasing function of μ for $\mu \geq \mu_{c_k,k}$. Hence, by Theorem 30, there exists a constant $c > 0$ such that a.a.s. the average degree of the $(k+1)$ -core of $\mathcal{G}(n, c/n)$ is $2k + o(1)$.

Let $\epsilon > 0$ be an arbitrarily small constant. For any $p \geq (c + \epsilon)/n$, a.a.s. the average degree of $\mathcal{G}(n, p)$ is greater than $2k + \sigma$, for some $\sigma(\epsilon) > 0$, by Theorem 30. Hence, we may assume that $p \leq (c + \epsilon)/n$, and it follows by Theorem 30 that the $(k + 1)$ -core of $\mathcal{G}(n, p)$ has average degree at most $2k + O(\epsilon)$. We will prove that for every $\epsilon > 0$ and every $p \leq (c + \epsilon)/n$, a.a.s. all subgraphs of $\mathcal{G}(n, p)$ have average degree at most $2k + O(\epsilon)$. Construct $\mathcal{G}(n, p)$ by exposing each non-edge in $\mathcal{G}(n, (c - \epsilon)/n)$ independently with probability p' , where p' satisfies

$$\frac{c - \epsilon}{n} + \left(1 - \frac{c - \epsilon}{n}\right) p' = p \leq \frac{c + \epsilon}{n}.$$

Then $p' = O(\epsilon/n)$. Thus, a.a.s. $\mathcal{G}(n, p)$ contains $O(\epsilon n)$ extra more edges than $\mathcal{G}(n, (c - \epsilon)/n)$. Let H be a densest subgraph of $\mathcal{G}(n, p)$. If the average degree of H is less than $3 \leq 2k$, we are done. Otherwise, by Lemma 29, a.a.s. $|V(H)| = \Omega(n)$. By Theorem 30, there exists $\sigma' > 0$ depending on ϵ such that the $(k + 1)$ -core of $\mathcal{G}(n, (c - \epsilon)/n)$ has average degree at most $2k - \sigma'$. By Theorem 31, a.a.s. the subgraph of $\mathcal{G}(n, (c - \epsilon)/n)$ induced by $V(H)$ has average degree at most $2k$. Adding $O(\epsilon n)$ edges to $V(H)$ will change its average degree by $O(\epsilon)$ because $|V(H)| = \Omega(n)$. Thus, the average degree of H in $\mathcal{G}(n, p)$ is at most $2k + O(\epsilon)$. This holds for every $\epsilon > 0$. Hence, a.a.s. this is no subgraph of $\mathcal{G}(n, p)$ whose average degree is more than $2k + o(1)$ for every $p \sim c/n$. \square

Proof of Theorem 4. Part (a) follows as a corollary of Theorem 5. Now we consider $p = \Theta(1/n)$. Assume $p \leq c/n$ for some constant $c > 0$. Then there exists $\sigma_c > 0$ such that a.a.s. the number of vertices in $G \sim \mathcal{G}(n, p)$ with degree 0 is at least $\sigma_c n$, whereas a.a.s. $m(G) = (1 + o(1))cn/2$. Hence, every forest contained in G has size at most $(1 - \sigma_c)n$. It follows then that $A(G) \geq m(G)/(1 - \sigma_c)n \geq (1 + \Theta(1))c/2$. Next, we prove that for all $c = c_n = \Theta(1)$, a.a.s. $A(\mathcal{G}(n, c/n))$ is concentrated on two bounded values. This directly implies that $A(\mathcal{G}(n, c/n))$ is bounded and thus, for every $c = \Theta(1)$, a.a.s. $A(\mathcal{G}(n, c/n)) = (1 + \Theta(1))c/2$.

To prove that $A(\mathcal{G}(n, c/n))$ is concentrated on two values, we consider two cases. If $\limsup_{n \rightarrow \infty} c \leq 1$, then all vertices of $\mathcal{G}(n, c/n)$ are contained in isolated trees except a set S of $o(n)$ vertices [9, Theorem 4b]. By Lemma 29, a.a.s. any subset $S' \subseteq S$ contains at most $1.1|S'|$ edges. Hence, a.a.s. there is no subgraph H of $\mathcal{G}(n, c/n)$, such that $|E(H)|/(|H| - 1) > 2$. It follows then that $A(\mathcal{G}(n, c/n)) \in \{1, 2\}$ in this case. Now we assume that $\liminf_{n \rightarrow \infty} c > 1$. We prove the a.a.s. two-value concentration of $A(\mathcal{G}(n, p))$ with $p \sim c/n$ for some constant $c > 1$ and our claim holds for all p in this range by the subsubsequence principle (see [16]). By Theorem 30, for every k , such that $c_k < c$, a.a.s. the average degree of the k -core of $A(\mathcal{G}(n, c/n))$ is $\mu_{c,k} f_{k-1}(\mu_{c,k})/f_k(\mu_{c,k}) + o(1)$. Let

$$K_c = \{k : c_k \leq c, \mu_{c,k} f_{k-1}(\mu_{c,k})/f_k(\mu_{c,k}) > 2(k - 1)\}.$$

Since $c > c_2 = 1$, a.a.s. there is a giant 2-core whose average degree is strictly greater than 2, and thus trivially $2 \in K_c$. So K_c is non-empty. It is well known that c_k is an increasing sequence of k and $c_k \rightarrow \infty$ as $k \rightarrow \infty$. Hence, K_c is finite. Let k_c be the largest integer in K_c . By Theorem 30, a.a.s. the $(k_c + 1)$ -core has average degree at most $2k_c + o(1)$ and the k_c -core has average degree strictly greater than $2(k_c - 1)$. By Corollary 32, a.a.s. there is no subgraph of $\mathcal{G}(n, c/n)$ whose average degree is more than $2k_c + o(1)$. Hence, the average degree of the densest subgraph is a.a.s. strictly greater than $2(k_c - 1)$ and at most $2k_c + o(1)$. It follows immediately by Theorem 8 that $A(\mathcal{G}(n, c/n)) \in \{k_c, k_c + 1\}$ for all $c = \Theta(1)$ such that $\liminf_{n \rightarrow \infty} c > 1$. Part (b) follows by defining $k_c = 1$ for all c such that $\limsup_{n \rightarrow \infty} c \leq 1$.

When $p = o(1/n)$, we have that $\mathcal{G}(n, p)$ is a.a.s. acyclic by [9] and nonempty. So a.a.s. $A(\mathcal{G}(n, p)) \leq 1$. □

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